This is a problem about powers and parity.

a. Consider the sequence \((a_0, a_1, a_2, \ldots) = (2, 2, 6, \ldots)\) defined by the formula

\[ a_n = (1 + \sqrt{2})^n + (1 - \sqrt{2})^n. \]

Find a simple recurrence relation that is satisfied by this sequence.

b. Prove that \([(1+\sqrt{2})^n] \equiv n \pmod{2}\) for all integers \(n > 0\).

c. Find a number \(x\) of the form \((p + \sqrt{q})/2\), where \(p\) and \(q\) are positive integers, such that \([x^n] \equiv n \pmod{2}\) for all integers \(n > 0\).

**Bonus problems**

50. Continuing exercise 22, consider the sum of all ways to decompose polygons into polygons:

\[ Q = \triangle + \square + \square + \square + \square + \square + \triangle + \square + \square + \square + \square + \square + \square + \square + \].

Find a symbolic equation for \(Q\) and use it to find a generating function for the number of ways to draw nonintersecting diagonals inside a convex \(n\)-gon. (Give a closed form for the generating function as a function of \(z\); you need not find a closed form for the coefficients.)

51. Prove that the product

\[ 2^{mn/2} \prod_{1 \leq j \leq m} \prod_{1 \leq k \leq n} \left( \cos^2 \left( \frac{j\pi}{m+1} \right) \right)^2 + \left( \cos^2 \left( \frac{k\pi}{n+1} \right) \right)^2 \]

is the generating function for tilings of an \(m \times n\) rectangle with dominoes. (There are \(mn\) factors, which we can imagine are written in the \(mn\) cells of the rectangle. If \(mn\) is odd, the middle factor is zero. The coefficient of \(\square^j \triangle^k\) is the number of ways to do the tiling with \(j\) vertical and \(k\) horizontal dominoes.) Hint: This is a difficult problem, really beyond the scope of this book. You may wish to simply verify the formula in the case \(m = 3, n = 4\).

52. Prove that the polynomials defined by the recurrence

\[ p_n(y) = (y - \frac{1}{4})^n - \sum_{k=0}^{n-1} \binom{2n}{2k} \left( \frac{-1}{4} \right)^{n-k} p_k(y), \quad \text{integer } n \geq 0, \]

have the form \(p_n(y) = \sum_{m=0}^{n} \left| \frac{n}{m} \right| y^m\), where \(\left| \frac{n}{m} \right|\) is a positive integer for \(1 \leq m \leq n\). Hint: This exercise is very instructive but not very easy.