Consider now a set of \( n \) vectors, each of which has \( n \) components. Let \( x_{ij} = x_i^{(j)} \) be the \( i \)th component of the vector \( x^{(j)} \), and let \( X = (x_{ij}) \). Then Eq. (17) can be written as

\[
\begin{pmatrix}
    x_1^{(1)} c_1 + \cdots + x_1^{(n)} c_n \\
    \vdots \\
    x_n^{(1)} c_1 + \cdots + x_n^{(n)} c_n
\end{pmatrix}
\begin{pmatrix}
    c_1 \\
    \vdots \\
    c_n
\end{pmatrix}
= Xc = 0.
\] (18)

If \( \det X \neq 0 \), then the only solution of Eq. (18) is \( c = 0 \), but if \( \det X = 0 \), there are nonzero solutions. Thus the set of vectors \( x^{(1)}, \ldots, x^{(n)} \) is linearly independent if and only if \( \det X \neq 0 \).

**Example 3**

Determine whether the vectors

\[
x^{(1)} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \quad x^{(2)} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \quad x^{(3)} = \begin{pmatrix} -4 \\ 1 \\ -11 \end{pmatrix}
\] (19)

are linearly independent or linearly dependent. If linearly dependent, find a linear relation among them.

To determine whether \( x^{(1)} \), \( x^{(2)} \), and \( x^{(3)} \) are linearly dependent we compute \( \det(x_{ij}) \), whose columns are the components of \( x^{(1)} \), \( x^{(2)} \), and \( x^{(3)} \), respectively. Thus

\[
\det(x_{ij}) = \begin{vmatrix} 1 & 2 & -4 \\ 2 & 1 & 1 \\ -1 & 3 & -11 \end{vmatrix},
\]

and an elementary calculation shows that it is zero. Thus \( x^{(1)} \), \( x^{(2)} \), and \( x^{(3)} \) are linearly dependent, and there are constants \( c_1 \), \( c_2 \), and \( c_3 \) such that

\[
c_1 x^{(1)} + c_2 x^{(2)} + c_3 x^{(3)} = 0. \] (20)

Equation (20) can also be written in the form

\[
\begin{pmatrix} 1 & 2 & -4 \\ 2 & 1 & 1 \\ -1 & 3 & -11 \end{pmatrix}
\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \] (21)

and solved by means of elementary row operations starting from the augmented matrix

\[
\begin{pmatrix} 1 & 2 & -4 & | & 0 \\ 2 & 1 & 1 & | & 0 \\ -1 & 3 & -11 & | & 0 \end{pmatrix}. \] (22)

We proceed as in Examples 1 and 2.

(a) Add \((-2)\) times the first row to the second row, and add the first row to the third row.

\[
\begin{pmatrix} 1 & 2 & -4 & | & 0 \\ 0 & -3 & 9 & | & 0 \\ 0 & 5 & -15 & | & 0 \end{pmatrix}
\]