Now setting \( \lambda = -1 \) in Eq. (31), we obtain
\[
\begin{pmatrix} 4 & -1 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\tag{36}
\]
Again we obtain a single condition on \( x_1 \) and \( x_2 \), namely, \( 4x_1 - x_2 = 0 \). Thus the eigenvector corresponding to the eigenvalue \( \lambda_2 = -1 \) is
\[
x^{(2)} = \begin{pmatrix} 1 \\ 4 \end{pmatrix},
\tag{37}
\]
or any nonzero multiple of this vector.

As Example 4 illustrates, eigenvectors are determined only up to an arbitrary nonzero multiplicative constant; if this constant is specified in some way, then the eigenvectors are said to be normalized. In Example 4, we set the constant equal to 1, but any other nonzero value could also have been used. Sometimes it is convenient to normalize an eigenvector \( \mathbf{x} \) by choosing the constant so that \( \langle \mathbf{x}, \mathbf{x} \rangle = 1 \).

Equation (27) is a polynomial equation of degree \( n \) in \( \lambda \), so there are \( n \) eigenvalues \( \lambda_1, \ldots, \lambda_n \), some of which may be repeated. If a given eigenvalue appears \( m \) times as a root of Eq. (27), then that eigenvalue is said to have multiplicity \( m \). Each eigenvalue has at least one associated eigenvector, and an eigenvalue of multiplicity \( m \) may have \( q \) linearly independent eigenvectors, where
\[
1 \le q \le m.
\tag{38}
\]
Examples show that \( q \) may be any integer in this interval. If all the eigenvalues of a matrix \( \mathbf{A} \) are simple (have multiplicity one), then it is possible to show that the \( n \) eigenvectors of \( \mathbf{A} \), one for each eigenvalue, are linearly independent. On the other hand, if \( \mathbf{A} \) has one or more repeated eigenvalues, then there may be fewer than \( n \) linearly independent eigenvectors associated with \( \mathbf{A} \), since for a repeated eigenvalue we may have \( q < m \). As we will see in Section 7.8, this fact may lead to complications later on in the solution of systems of differential equations.

\[\text{Example 5}\]

Find the eigenvalues and eigenvectors of the matrix
\[
\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.
\tag{39}
\]
The eigenvalues \( \lambda \) and eigenvectors \( \mathbf{x} \) satisfy the equation \((\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = \mathbf{0}\), or
\[
\begin{pmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
\tag{40}
\]
The eigenvalues are the roots of the equation
\[
\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = -\lambda^3 + 3\lambda + 2 = 0.
\tag{41}
\]