The roots of Eq. (41) are \( \lambda_1 = 2, \lambda_2 = -1, \) and \( \lambda_3 = -1. \) Thus 2 is a simple eigenvalue, and \(-1\) is an eigenvalue of multiplicity 2.

To find the eigenvector \( \mathbf{x}^{(1)} \) corresponding to the eigenvalue \( \lambda_1 \) we substitute \( \lambda = 2 \) in Eq. (40). This gives the system

\[
\begin{pmatrix}
-2 & 1 & 1 \\
1 & -2 & 1 \\
1 & 1 & -2
\end{pmatrix}
\begin{pmatrix}
 x_1 \\
 x_2 \\
 x_3
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}.
\] (42)

We can reduce this to the equivalent system

\[
\begin{pmatrix}
2 & -1 & -1 \\
0 & 1 & -1 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
 x_1 \\
 x_2 \\
 x_3
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\] (43)

by elementary row operations. Solving this system we obtain the eigenvector

\[
\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.
\] (44)

For \( \lambda = -1 \), Eqs. (40) reduce immediately to the single equation

\[
x_1 + x_2 + x_3 = 0.
\] (45)

Thus values for two of the quantities \( x_1, x_2, x_3 \) can be chosen arbitrarily and the third is determined from Eq. (45). For example, if \( x_1 = 1 \) and \( x_2 = 0 \), then \( x_3 = -1 \), and

\[
\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}
\] (46)

is an eigenvector. Any nonzero multiple of \( \mathbf{x}^{(2)} \) is also an eigenvector, but a second independent eigenvector can be found by making another choice of \( x_1 \) and \( x_2 \); for instance, \( x_1 = 0 \) and \( x_2 = 1 \). Again \( x_3 = -1 \) and

\[
\mathbf{x}^{(3)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}
\] (47)

is an eigenvector linearly independent of \( \mathbf{x}^{(2)} \). Therefore in this example two linearly independent eigenvectors are associated with the double eigenvalue.

An important special class of matrices, called \textit{self-adjoint} or \textit{Hermitian} matrices, are those for which \( \mathbf{A}^* = \mathbf{A} \); that is, \( \overline{a}_{ij} = a_{ij} \). Hermitian matrices include as a subclass real symmetric matrices, that is, matrices that have real elements and for which \( \mathbf{A}^T = \mathbf{A} \).

The eigenvalues and eigenvectors of Hermitian matrices always have the following useful properties:

1. All eigenvalues are real.
2. There always exists a full set of \( n \) linearly independent eigenvectors, regardless of the multiplicities of the eigenvalues.