13. \( x^{(1)}(t) = (2 \sin t, \sin t), \quad x^{(2)}(t) = (\sin t, 2 \sin t) \)

14. Let

\[
x^{(1)}(t) = \begin{pmatrix} e^t \\ te^t \end{pmatrix}, \quad x^{(2)}(t) = \begin{pmatrix} 1 \\ t \end{pmatrix}.
\]

Show that \( x^{(1)}(t) \) and \( x^{(2)}(t) \) are linearly dependent at each point in the interval \( 0 \leq t \leq 1 \). Nevertheless, show that \( x^{(1)}(t) \) and \( x^{(2)}(t) \) are linearly independent on \( 0 \leq t \leq 1 \).

In each of Problems 15 through 24 find all eigenvalues and eigenvectors of the given matrix.

15. \[
\begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}
\]

16. \[
\begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}
\]

17. \[
\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}
\]

18. \[
\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}
\]

19. \[
\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}
\]

20. \[
\begin{pmatrix} -3 & 3/4 \\ -5 & 1 \end{pmatrix}
\]

21. \[
\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix}
\]

22. \[
\begin{pmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -2 & -4 & -1 \end{pmatrix}
\]

23. \[
\begin{pmatrix} 11/9 & -2/9 & 8/9 \\ -2/9 & 2/9 & 10/9 \\ 8/9 & 10/9 & 5/9 \end{pmatrix}
\]

24. \[
\begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}
\]

Problems 25 through 29 deal with the problem of solving \( Ax = b \) when \( \det A = 0 \).

25. Suppose that, for a given matrix \( A \), there is a nonzero vector \( x \) such that \( Ax = 0 \). Show that there is also a nonzero vector \( y \) such that \( A^*y = 0 \).

26. Show that \( (Ax, y) = (x, A^*y) \) for any vectors \( x \) and \( y \).

27. Suppose that \( \det A = 0 \) and that \( Ax = b \) has solutions. Show that \( (b, y) = 0 \), where \( y \) is any solution of \( A^*y = 0 \). Verify that this statement is true for the set of equations in Example 2.

\textit{Hint:} Use the result of Problem 26.

28. Suppose that \( \det A = 0 \), and that \( x = x^{(0)} \) is a solution of \( Ax = b \). Show that if \( \xi \) is a solution of \( A\xi = 0 \) and \( \alpha \) is any constant, then \( x = x^{(0)} + \alpha \xi \) is also a solution of \( Ax = b \).

29. Suppose that \( \det A = 0 \) and that \( y \) is a solution of \( A^*y = 0 \). Show that if \( (b, y) = 0 \) for every such \( y \), then \( Ax = b \) has solutions. Note that this is the converse of Problem 27; the form of the solution is given by Problem 28.

30. Prove that \( \lambda = 0 \) is an eigenvalue of \( A \) if and only if \( A \) is singular.

31. Prove that if \( A \) is Hermitian, then \( (Ax, y) = (x, Ay) \), where \( x \) and \( y \) are any vectors.

32. In this problem we show that the eigenvalues of a Hermitian matrix \( A \) are real. Let \( x \) be an eigenvector corresponding to the eigenvalue \( \lambda \).

(a) Show that \( (Ax, x) = (x, Ax) \). \textit{Hint:} See Problem 31.

(b) Show that \( \lambda(x, x) = \lambda(x, x) \). \textit{Hint:} Recall that \( Ax = \lambda x \).

(c) Show that \( \lambda = \bar{\lambda} \); that is, the eigenvalue \( \lambda \) is real.

33. Show that if \( \lambda_1 \) and \( \lambda_2 \) are eigenvalues of a Hermitian matrix \( A \), and if \( \lambda_1 \neq \lambda_2 \), then the corresponding eigenvectors \( x^{(1)} \) and \( x^{(2)} \) are orthogonal.

\textit{Hint:} Use the results of Problems 31 and 32 to show that \( (\lambda_1 - \lambda_2)(x^{(1)}, x^{(2)}) = 0 \).