The mean of a random variable turns out to be more meaningful in applications than the other kinds of averages, so we shall largely forget about medians and modes from now on. We will use the terms “expected value,” “mean,” and “average” almost interchangeably in the rest of this chapter.

If $X$ and $Y$ are any two random variables defined on the same probability space, then $X + Y$ is also a random variable on that space. By formula (8.9), the average of their sum is the sum of their averages:

$$E(X+Y) = \sum_{\omega \in \Omega} (X(\omega) + Y(\omega)) \cdot \Pr(\omega) = EX + EY.$$  \hspace{1cm} (8.10)

Similarly, if $\alpha$ is any constant we have the simple rule

$$E(\alpha X) = \alpha EX.$$  \hspace{1cm} (8.11)

But the corresponding rule for multiplication of random variables is more complicated in general; the expected value is defined as a sum over elementary events, and sums of products don’t often have a simple form. In spite of this difficulty, there is a very nice formula for the mean of a product in the special case that the random variables are independent:

$$E(XY) = \langle EX \rangle \langle EY \rangle, \quad \text{if } X \text{ and } Y \text{ are independent.}$$  \hspace{1cm} (8.12)

We can prove this by the distributive law for products,

$$E(XY) = \sum_{\omega \in \Omega} X(\omega)Y(\omega) \cdot \Pr(\omega)$$

$$= \sum_{x \in \Omega} x \cdot \sum_{y \in \Omega} y \cdot \Pr(X=x \text{ and } Y=y)$$

$$= \sum_{x \in \Omega} x \cdot \sum_{y \in \Omega} \Pr(Y=y) \cdot \Pr(X=x)$$

$$= \sum_{x \in \Omega} x \cdot \Pr(X=x) \cdot \sum_{y \in \Omega} y \cdot \Pr(Y=y)$$

$$= \langle EX \rangle \langle EY \rangle.$$  \hspace{1cm} (8.12)

For example, we know that $S = S_1 + S_2$ and $P = S_1 S_2$, when $S_1$ and $S_2$ are the numbers of spots on the first and second of a pair of random dice. We have $\langle S \rangle = \langle S_1 \rangle + \langle S_2 \rangle = \frac{7}{2} + \frac{7}{2} = 7$, hence $\langle S \rangle = 7$; furthermore $S_1$ and $S_2$ are independent, so $\langle P \rangle = \frac{7}{2} \cdot \frac{7}{2} = \frac{49}{4}$, as claimed earlier. We also have $\langle S + P \rangle = \langle S \rangle + \langle P \rangle = 7 + \frac{49}{4}$.

But $S$ and $P$ are not independent, so we cannot assert that $\langle SP \rangle = \langle S \rangle \cdot \langle P \rangle$. In fact, the expected value of $SP$ turns out to equal $\frac{637}{6}$ in distribution $\Pr_{11}$, 112 (exactly) in distribution $\Pr_{00}$. \hspace{1cm} I get it: On average, “average” means “mean.”