Theorem 7.4.1 we reach the conclusion that if \( x^{(1)}, \ldots, x^{(k)} \) are solutions of Eq. (3) then
\[
x = c_1 x^{(1)}(t) + \cdots + c_k x^{(k)}(t)
\] (5)
is also a solution for any constants \( c_1, \ldots, c_k \). As an example, it can be verified that
\[
x^{(1)}(t) = \left( \begin{array}{c} e^{3t} \\ 2e^{3t} \end{array} \right), \quad x^{(2)}(t) = \left( \begin{array}{c} e^{-t} \\ -2e^{-t} \end{array} \right)
\] (6)
satisfy the equation
\[
x' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} x.
\] (7)
According to Theorem 7.4.1
\[
x = c_1 \frac{1}{2} e^{3t} + c_2 \frac{1}{-2} e^{-t}
\]
also satisfies Eq. (7).

As we indicated previously, by repeatedly applying Theorem 7.4.1, it follows that every finite linear combination of solutions of Eq. (3) is also a solution. The question now arises as to whether all solutions of Eq. (3) can be found in this way. By analogy with previous cases it is reasonable to expect that for a system of the form (3) of \( n \)th order it is sufficient to form linear combinations of \( n \) properly chosen solutions. Therefore let \( x^{(1)}, \ldots, x^{(n)} \) be \( n \) solutions of the \( n \)th order system (3), and consider the matrix \( X(t) \) whose columns are the vectors \( x^{(1)}(t), \ldots, x^{(n)}(t) \):
\[
X(t) = \begin{pmatrix}
x^{(1)}_1(t) & \cdots & x^{(1)}_n(t) \\
\vdots & \ddots & \vdots \\
x^{(n)}_1(t) & \cdots & x^{(n)}_n(t)
\end{pmatrix}.
\] (9)
Recall from Section 7.3 that the columns of \( X(t) \) are linearly independent for a given value of \( t \) if and only if \( \det X \neq 0 \) for that value of \( t \). This determinant is called the Wronskian of the \( n \) solutions \( x^{(1)}, \ldots, x^{(n)} \) and is also denoted by \( W[x^{(1)}, \ldots, x^{(n)}] \); that is,
\[
W[x^{(1)}, \ldots, x^{(n)}](t) = \det X(t).
\] (10)
The solutions \( x^{(1)}, \ldots, x^{(n)} \) are then linearly independent at a point if and only if \( W[x^{(1)}, \ldots, x^{(n)}] \) is not zero there.

**Theorem 7.4.2** If the vector functions \( x^{(1)}, \ldots, x^{(n)} \) are linearly independent solutions of the system (3) for each point in the interval \( \alpha < t < \beta \), then each solution \( x = \Phi(t) \) of the system (3) can be expressed as a linear combination of \( x^{(1)}, \ldots, x^{(n)} \),
\[
\Phi(t) = c_1 x^{(1)}(t) + \cdots + c_n x^{(n)}(t),
\] (11)
in exactly one way.