Before proving Theorem 7.4.2, note that according to Theorem 7.4.1 all expressions of the form (11) are solutions of the system (3), while by Theorem 7.4.2 all solutions of Eq. (3) can be written in the form (11). If the constants $c_1, \ldots, c_n$ are thought of as arbitrary, then Eq. (11) includes all solutions of the system (3) and it is customary to call it the general solution. Any set of solutions $x^{(1)}, \ldots, x^{(n)}$ of Eq. (3) which is linearly independent at each point in the interval $\alpha < t < \beta$, is said to be a fundamental set of solutions for that interval.

To prove Theorem 7.4.2, we will show, given any solution $\phi$ of Eq. (3) that $\phi(t) = c_1 x^{(1)}(t) + \cdots + c_n x^{(n)}(t)$ for suitable values of $c_1, \ldots, c_n$. Let $t = t_0$ be some point in the interval $\alpha < t < \beta$ and let $\xi = \phi(t_0)$. We now wish to determine whether there is any solution of the form $x = c_1 x^{(1)}(t) + \cdots + c_n x^{(n)}(t)$ that also satisfies the same initial condition $x(t_0) = \xi$. That is, we wish to know whether there are values of $c_1, \ldots, c_n$ such that

$$c_1 x^{(1)}(t_0) + \cdots + c_n x^{(n)}(t_0) = \xi,$$

or in scalar form

$$c_1 x_{11}(t_0) + \cdots + c_n x_{1n}(t_0) = \xi_1,$$

$$\vdots$$

$$c_1 x_{n1}(t_0) + \cdots + c_n x_{nn}(t_0) = \xi_n.$$  

The necessary and sufficient condition that Eqs. (13) possess a unique solution $c_1, \ldots, c_n$ is precisely the nonvanishing of the determinant of coefficients, which is the Wronskian $W[x^{(1)}, \ldots, x^{(n)}]$ evaluated at $t = t_0$. The hypothesis that $x^{(1)}, \ldots, x^{(n)}$ are linearly independent throughout $\alpha < t < \beta$ guarantees that $W[x^{(1)}, \ldots, x^{(n)}]$ is not zero at $t = t_0$, and therefore there is a (unique) solution of Eq. (3) of the form $x = c_1 x^{(1)}(t) + \cdots + c_n x^{(n)}(t)$ that also satisfies the initial condition (12). By the uniqueness part of Theorem 7.1.2 this solution is identical to $\phi(t)$, and hence $\phi(t) = c_1 x^{(1)}(t) + \cdots + c_n x^{(n)}(t)$, as was to be proved.

**Theorem 7.4.3** If $x^{(1)}, \ldots, x^{(n)}$ are solutions of Eq. (3) on the interval $\alpha < t < \beta$, then in this interval $W[x^{(1)}, \ldots, x^{(n)}]$ either is identically zero or else never vanishes.

The significance of Theorem 7.4.3 lies in the fact that it relieves us of the necessity of examining $W[x^{(1)}, \ldots, x^{(n)}]$ at all points in the interval of interest, and enables us to determine whether $x^{(1)}, \ldots, x^{(n)}$ form a fundamental set of solutions merely by evaluating their Wronskian at any convenient point in the interval.

Theorem 7.4.3 is proved by first establishing that the Wronskian of $x^{(1)}, \ldots, x^{(n)}$ satisfies the differential equation (see Problem 2)

$$\frac{dW}{dt} = (p_{11} + p_{22} + \cdots + p_{nn})W.$$  

Hence $W$ is an exponential function, and the conclusion of the theorem follows immediately. The expression for $W$ obtained by solving Eq. (14) is known as Abel’s formula; note the analogy with Eq. (8) of Section 3.3.

Alternatively, Theorem 7.4.3 can be established by showing that if $n$ solutions $x^{(1)}, \ldots, x^{(n)}$ of Eq. (3) are linearly dependent at one point $t = t_0$, then they must