be linearly dependent at each point in \( \alpha < t < \beta \) (see Problem 8). Consequently, if \( x^{(1)}, \ldots, x^{(n)} \) are linearly independent at one point, they must be linearly independent at each point in the interval.

The next theorem states that the system (3) always has at least one fundamental set of solutions.

**Theorem 7.4.4**  
Let

\[
e^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \quad e^{(n)} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix};
\]

further let \( x^{(1)}, \ldots, x^{(n)} \) be the solutions of the system (3) satisfying the initial conditions

\[
x^{(1)}(t_0) = e^{(1)}, \ldots, x^{(n)}(t_0) = e^{(n)},
\]

respectively, where \( t_0 \) is any point in \( \alpha < t < \beta \). Then \( x^{(1)}, \ldots, x^{(n)} \) form a fundamental set of solutions of the system (3).

To prove this theorem, note that the existence and uniqueness of the solutions \( x^{(1)}, \ldots, x^{(n)} \) mentioned in Theorem 7.4.4 are assured by Theorem 7.1.2. It is not hard to see that the Wronskian of these solutions is equal to 1 when \( t = t_0 \); therefore \( x^{(1)}, \ldots, x^{(n)} \) are a fundamental set of solutions.

Once one fundamental set of solutions has been found, other sets can be generated by forming (independent) linear combinations of the first set. For theoretical purposes the set given by Theorem 7.4.4 is usually the simplest.

To summarize, any set of \( n \) linearly independent solutions of the system (3) constitutes a fundamental set of solutions. Under the conditions given in this section, such fundamental sets always exist, and every solution of the system (3) can be represented as a linear combination of any fundamental set of solutions.

**PROBLEMS**

1. Using matrix algebra, prove the statement following Theorem 7.4.1 for an arbitrary value of the integer \( k \).

2. In this problem we outline a proof of Theorem 7.4.3 in the case \( n = 2 \). Let \( x^{(1)} \) and \( x^{(2)} \) be solutions of Eq. (3) for \( \alpha < t < \beta \), and let \( W \) be the Wronskian of \( x^{(1)} \) and \( x^{(2)} \).
   (a) Show that
   \[
   \frac{dW}{dt} = \begin{vmatrix} \frac{dx^{(1)}}{dt} & \frac{dx^{(2)}}{dt} \\ \frac{dx^{(1)}}{dt} & \frac{dx^{(2)}}{dt} \end{vmatrix} + \begin{vmatrix} x^{(1)}_1 & x^{(2)}_1 \\ x^{(1)}_2 & x^{(2)}_2 \end{vmatrix} \cdot \begin{vmatrix} \frac{dx^{(1)}}{dt} & \frac{dx^{(2)}}{dt} \\ \frac{dx^{(1)}}{dt} & \frac{dx^{(2)}}{dt} \end{vmatrix}.
   \]
   (b) Using Eq. (3), show that
   \[
   \frac{dW}{dt} = (p_{11} + p_{22})W.
   \]
   (c) Find \( W(t) \) by solving the differential equation obtained in part (b). Use this expression to obtain the conclusion stated in Theorem 7.4.3.