1) The operator

\[ A : V_{2,\beta,\gamma}^l(\mathcal{G}) \rightarrow V_{2,\beta,\gamma}^{l-2m}(\mathcal{G}) \times \prod_{k=1}^{m} V_{2,\beta,\gamma}^{l-k+1/2}(\partial \mathcal{G}) \]

of the Dirichlet problem (10.5.10), (10.5.11) realizes an isomorphism.

2) If \( u \in W_2^{2m}(\mathcal{G}) \) is a generalized solution of problem (10.5.10), (10.5.11) and \( f \in V_{2,\beta,\gamma}^{l-2m}(\mathcal{G}), g \in V_{2,\beta,\gamma}^{l-k+1/2}(\partial \mathcal{G}) \), then \( u \in V_{2,\beta,\gamma}^l(\mathcal{G}) \).

**Remark 10.5.2.** Everything said before about the Dirichlet problem for \( 2m < n - 1 \) is also true for general boundary value problems (10.1.2), (10.1.3) with \( J = 0 \) (i.e., there do not appear unknowns \( u^{(s)} \) on the boundary of \( \mathcal{G} \) which can be written in the variational form. We suppose that the orders of the differential operators \( B_{s,k} \) are less than \( 2m \) and there exists a sesquilinear form \( a(\cdot, \cdot) \) on \( W_2^{2m}(\mathcal{G}) \times W_2^{2m}(\mathcal{G}) \) such that the Green formula

\[ a(u, v) = \int_{\mathcal{G}} Lu \cdot \overline{v} \, dx + \sum_{(s,k) \in I_1 \Gamma_s} \int_{\Gamma_s} B_{s,k} u \cdot \overline{T_{s,k} v} \, d\sigma + \sum_{(s,k) \in I_2 \Gamma_s} \int_{\Gamma_s} S_{s,k} u \cdot B_{s,k} \overline{v} \, d\sigma \]

is valid for all functions \( u, v \in W_2^{2m}(\mathcal{G}) \) with compact support. Here the set \( I_1 \) contains all pairs \( (s,k) \) such that \( m \leq \text{ord} B_{s,k} \leq 2m - 1 \), while the set \( I_2 \) contains all pairs \( (s,k) \) such that \( \text{ord} B_{s,k} < m \). As for the Dirichlet problem, we assume that Gårding's inequality

\[ |\text{Re} a(u, u)| \geq c \|u\|^2_{W_2^{2m}(\mathcal{G})} \]

is satisfied for all \( u \in W_2^{2m}(\mathcal{G}) \) such that \( B_{s,k} u = 0 \) on \( \Gamma_s \) for \( (s,k) \in I_2 \), i.e., the sesquilinear form \( a(\cdot, \cdot) \) is \( V \)-elliptic (see Definition 4.3.1). Furthermore, we assume that \( \beta \in (l - (n - 1)/2, l - 2m + (n - 1)/2) \). Then assertions 1) and 2) of Theorem 10.5.3 (in the case \( 2m < n - 1 \)) are valid for the boundary value problem (10.1.2), (10.1.3).

### 10.5.3. Boundary value problems in the exterior of a "cusp", "paraboloid", or "infinite funnel".

Let \( \mathcal{G} \) be a domain in \( \mathbb{R}^n \) with a compact closure \( \overline{\mathcal{G}} \) and boundary \( \partial \mathcal{G} \). We suppose that the surface \( \partial \mathcal{G} \) is everywhere smooth (of class \( C^1 \)), except at the point 0 which coincides with the origin.

For an arbitrary point \( x = (x', x_n) \in \mathbb{R}^n \) we set \( \rho = |x|, \omega = x'/|x|, r = |x'| \). The coordinates \( x' = (x_1, \ldots, x_{n-1}) \) are also local coordinates in a small neighbourhood of the north pole \( N = (0, \ldots, 0, 1) \) on the sphere \( S^{n-1} \). We denote by \( \tilde{\Omega} \) a domain in \( \mathbb{R}^{n-1} \) with a smooth boundary and assume that

\[ \partial B_\rho \setminus \mathcal{G} = \{ x \in \mathbb{R}^n : |x| = \rho, x'/\rho \in \varphi(\log \rho^{-1}) \tilde{\Omega} \} , \]

where \( B_\rho = \{ x \in \mathbb{R}^n : |x| < \rho \} \) and \( \varphi \) is a function satisfying condition (10.1.1). By \( V_{2,\beta}^l(\mathcal{G}) \) we denote the weighted Sobolev space with the norm

\[ \|u\|_{V_{2,\beta,\gamma}^l(\mathcal{G})} = \left( \sum_{|\alpha| \leq l} \int_{\mathcal{G}} \rho^{2(\gamma-l+|\alpha|)} \vartheta^{2(\beta-l+|\alpha|)} |D_\mathcal{G}^\alpha u|^2 \, dx \right)^{1/2} . \]

Here \( \theta \) is the angle between the vector 0x and the \( x_n \)-axis. For \( l \geq 1 \) we denote the corresponding trace space by \( V_{2,\beta,\gamma}^{l-1/2}(\partial \mathcal{G}) \).

Let \( \mathcal{G} \) be the image of the domain \( \mathcal{G} \) under the mapping \( x \rightarrow (\omega, t) \), where \( t = \log |x|^{-1} \) and \( \omega = x'/|x| \). This domain satisfies the conditions of the previous subsections.
The coordinate transformation $x \rightarrow (\omega, t)$ takes the space $\mathcal{V}_{\beta, \gamma}(\mathfrak{G})$ into the space $\mathcal{V}_{2, \beta, \gamma - n/2}(\mathcal{G})$. Hence from Theorems 10.5.1 - 10.5.3 for the boundary value problems in the domain $\mathcal{G}$ we obtain the same results for problems in the domain $\mathfrak{G}$. We give an example.

Let the boundary value problem
\begin{align}
L(x, \partial_x) u &= f \quad \text{in } \mathfrak{G}, \\
B_k(x, \partial_x) u &= g_k \quad \text{on } \partial \mathcal{G}, \quad k = 1, \ldots, m,
\end{align}
be given. We assume that the coefficients of the operator $L$ are smooth in $\mathbb{R}^n$, while those of the operators $B_k$ are smooth outside the origin in a neighbourhood of $\partial \mathcal{G}$. Moreover, we suppose that the operators $B_k$ have the representation
\[ B_k(x, \partial_x) = r^{-\mu_k} \sum_{|\alpha| + j \leq \mu_k} b_{k; \alpha, j}(x'/\varphi(\log \rho^{-1}))(r \partial_r)^\alpha (r \partial_\rho)^j \]
in a neighbourhood of the point 0, where $\mu_k < 2m$ and $b_{k; \alpha, j}$ are smooth functions on $\partial \mathcal{G}$, and that the operator $A$ of problem (10.5.17), (10.5.18) is V-elliptic (see Remark 10.5.2). This implies the unique solvability of problem (10.5.17), (10.5.18) in the space $W^m_2(\mathfrak{G})$ for $g_k = 0$. By Hardy’s inequality, the last space coincides with $V^m_{2,0,0}(\mathfrak{G})$ if $2m < n - 1$. Therefore, the condition of V-ellipticity of the operator $A = (L, B_k)$ turns into the condition of V-ellipticity of the operator $e^{t(m-n/2)}(\bar{L}, \bar{B}_k)e^{-t(m-n/2)}$, where $\bar{L}$, $\bar{B}_k$ are the operators obtained from $\rho^{2m}L$, $\rho^{\mu_k}B_k$ by the coordinate change $x \rightarrow (\omega, t)$, $t = \log \rho^{-1}$.

Now we can apply the result given in Remark 10.5.2 (taking into account Remark 10.5.1). In our case the numbers $\gamma_-, \gamma_+$ occurring in Theorem 10.5.3 can be computed explicitly.

Let $\bar{L}(\infty)$ be the operator obtained from $\bar{L}$ replacing the coefficients by their limits for $t \rightarrow \infty$. It can be directly verified that the inverse substitution $(\omega, t) \rightarrow x$ takes the operator $\bar{L}(\infty)$ into the operator $\rho^{2m}L^0(0, \partial_x)$, where $L^0$ is the principal part of $L$. This circumstance allows us to find all the eigenvalues of the pencil $\mathfrak{A}(\lambda)$ generated by the operator $\bar{L}(\infty)$ and defined on the space $V^m_{2,\beta}(S^{n-1})$, $\beta \in (2m - (n - 1)/2, (n - 1)/2)$. For this it is sufficient to find all nontrivial solutions of the equation $L^0(0, \partial_x)u = 0$ outside the $x_n$-axis having the form $\rho^\lambda \Phi_\lambda(\omega)$. Since the function $\Phi_\lambda$ satisfies a homogeneous elliptic equation with coefficients from $C^\infty(S^{n-1})$ on the sphere $S^{n-1}$ outside the north pole $N$, it follows that it is either smooth on $S^{n-1}$ or has a power-like singularity at the point $N$ of order $2m - n - k$, $k = -1, 0, 1, \ldots$. The latter is not possible since $\Phi_\lambda \in V^m_{2,\beta}(S^{n-1})$ and $\beta < (n - 1)/2$. Hence we may assume that $L^0(0, \partial_x)u = 0$ for $|x| > 0$. The solutions of this equation are either homogeneous polynomials of degrees $\lambda = 0, 1, \ldots$ or linear combinations of the derivatives of the fundamental solution of the operator $L^0(0, \partial_x)$. In this case, $\lambda = 2m - n, 2m - n - 1, \ldots$. Due to the fact that $\beta > 2m - (n - 1)/2$, the restrictions of all these solutions to the sphere $S^{n-1}$ belong to $V^m_{2,\beta}(S^{n-1})$.

Thus, the eigenvalues of the pencil $\mathfrak{A}(\lambda)$ are $2m - n, 2m - n - 1, \ldots$ and $0, 1, 2, \ldots$. Consequently, the eigenvalues of the pencil generated by the operator $e^{t(m-n/2)}\bar{L}e^{-t(m-n/2)}$ are $\pm(n/2 - m), \pm(n/2 - m + 1), \ldots$. Therefore, the largest strip in the $\lambda$-plane which is free of eigenvalues of the pencil generated by the operator $e^{t(m-n/2)}\bar{L}e^{-t(m-n/2)}$ and contains the imaginary axis has the form $|\text{Re } \lambda| < n/2 - m$. 