Similarly, corresponding to \( r_2 = -1 \), we find that \( \xi_2 = -2\xi_1 \), so the eigenvector is
\[
\xi^{(2)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.
\] (10)

The corresponding solutions of the differential equation are
\[
x^{(1)}(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}, \quad x^{(2)}(t) = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}.
\] (11)

The Wronskian of these solutions is
\[
W[x^{(1)}, x^{(2)}](t) = \begin{vmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{vmatrix} = -4e^{2t},
\] (12)
which is never zero. Hence the solutions \( x^{(1)} \) and \( x^{(2)} \) form a fundamental set, and the general solution of the system (5) is
\[
x = c_1 x^{(1)}(t) + c_2 x^{(2)}(t)
\] (13)
where \( c_1 \) and \( c_2 \) are arbitrary constants.

To visualize the solution (13) it is helpful to consider its graph in the \( x_1 x_2 \) plane for various values of the constants \( c_1 \) and \( c_2 \). We start with \( x = c_1 x^{(1)}(t) \), or in scalar form
\[
x_1 = c_1 e^{3t}, \quad x_2 = 2c_1 e^{3t}.
\]
By eliminating \( t \) between these two equations, we see that this solution lies on the straight line \( x_2 = 2x_1 \); see Figure 7.5.2a. This is the line through the origin in the direction of the eigenvector \( \xi^{(1)} \). If we look on the solution as the trajectory of a moving particle, then the particle is in the first quadrant when \( c_1 > 0 \) and in the third quadrant when \( c_1 < 0 \). In either case the particle departs from the origin as \( t \) increases. Next consider \( x = c_2 x^{(2)}(t) \), or
\[
x_1 = c_2 e^{-t}, \quad x_2 = -2c_2 e^{-t}.
\]
This solution lies on the line \( x_2 = -2x_1 \), whose direction is determined by the eigenvector \( \xi^{(2)} \). The solution is in the fourth quadrant when \( c_2 > 0 \) and in the second quadrant when \( c_2 < 0 \), as shown in Figure 7.5.2a. In both cases the particle moves toward the origin as \( t \) increases. The solution (13) is a combination of \( x^{(1)}(t) \) and \( x^{(2)}(t) \). For large \( t \) the term \( c_1 x^{(1)}(t) \) is dominant and the term \( c_2 x^{(2)}(t) \) becomes negligible. Thus all solutions for which \( c_1 \neq 0 \) are asymptotic to the line \( x_2 = 2x_1 \) as \( t \to \infty \). Similarly, all solutions for which \( c_2 \neq 0 \) are asymptotic to the line \( x_2 = -2x_1 \) as \( t \to -\infty \). The graphs of several solutions are shown in Figure 7.5.2a. The pattern of trajectories in this figure is typical of all second order systems \( x = Ax \) for which the eigenvalues are real and of opposite signs. The origin is called a saddle point in this case. Saddle points are always unstable because almost all trajectories depart from them as \( t \) increases.

In the preceding paragraph we have described how to draw by hand a qualitatively correct sketch of the trajectories of a system such as Eq. (5) once the eigenvalues and eigenvectors have been determined. However, to produce a detailed and accurate drawing, such as Figure 7.5.2a and other figures that appear later in this chapter, a computer is extremely helpful, if not indispensable.