variance by analyzing the mutual dependencies among them:

\[
E(F^2_n) = E\left( \left( \sum_{k=1}^{n} F_{n,k} \right)^2 \right) = E\left( \sum_{j=1}^{n} \sum_{k=1}^{n} F_{n,j} F_{n,k} \right)
\]

\[
= \sum_{j=1}^{n} \sum_{k=1}^{n} E(F_{n,j} F_{n,k}) = \sum_{i \leq k \leq n} E(F^2_{n,k}) + 2 \sum_{1 \leq j < k \leq n} E(F_{n,j} F_{n,k})
\]

(We used a similar trick when we derived (2.33) in Chapter 2.) Now \( F^2_{n,k} = F_{n,k} \), since \( F_{n,k} \) is either 0 or 1; hence \( E(F^2_{n,k}) = E(F_{n,k}) = 1/n \) as before. And if \( j < k \) we have \( E(F_{n,j} F_{n,k}) = \Pr(\pi \text{ has both } j \text{ and } k \text{ as fixed points}) = (n-2)!/n! = 1/n(n-1) \). Therefore

\[
E(F^2_n) = \frac{n}{n} + \frac{n}{n(n-1)} = 2, \quad \text{for } n \geq 2.
\] (8.24)

(As a check when \( n = 3 \), we have \( \frac{2}{6} 0^2 + \frac{3}{6} 1^2 + \frac{1}{6} 2^2 + \frac{1}{6} 3^2 = 2 \).) The variance is \( E(F^2_n) - (E(F_n))^2 = 1 \), so the standard deviation (like the mean) is \( 1 \). “A random permutation of \( n \) elements has \( 1 \) fixed points.”

### 8.3 Probability Generating Functions

If \( X \) is a random variable that takes only nonnegative integer values, we can capture its probability distribution nicely by using the techniques of Chapter 7. The probability generating function or pgf of \( X \) is

\[
G_X(z) = \sum_{k \geq 0} \Pr(X = k) z^k.
\] (8.25)

This power series in \( z \) contains all the information about the random variable \( X \). We can also express it in two other ways:

\[
G_X(z) = \sum_{\omega \in \Omega} \Pr(\omega) z^{X(\omega)} = E(z^X).
\] (8.26)

The coefficients of \( G_X(z) \) are nonnegative, and they sum to 1; the latter condition can be written

\[
G_X(1) = 1.
\] (8.27)

Conversely, any power series \( G(z) \) with nonnegative coefficients and with \( G(1) = 1 \) is the pgf of some random variable.