The nicest thing about pgf’s is that they usually simplify the computation of means and variances. For example, the mean is easily expressed:

\[ EX = \sum_{k \geq 0} k \cdot \Pr(X = k) \]
\[ = \sum_{k \geq 0} \Pr(X = k) \cdot k z^{k-1} \bigg|_{z=1} \]
\[ = G_X(1). \quad (8.28) \]

We simply differentiate the pgf with respect to \( z \) and set \( z = 1 \).

The variance is only slightly more complicated:

\[ E(X^2) = \sum_{k \geq 0} k^2 \cdot \Pr(X = k) \]
\[ = \sum_{k \geq 0} \Pr(X = k) \cdot (k(k-1) z^{k-2} + k z^{k-1}) \bigg|_{z=1} = G''_X(1) + G'_X(1). \]

Therefore

\[ VX = G''_X(1) + G'_X(1) - G'_X(1)^2. \quad (8.29) \]

Equations (8.28) and (8.29) tell us that we can compute the mean and variance if we can compute the values of two derivatives, \( G''_X(1) \) and \( G'_X(1) \). We don’t have to know a closed form for the probabilities; we don’t even have to know a closed form for \( G_X(z) \) itself.

It is convenient to write

\[ \text{Mean}(G) = G'(1), \quad (8.30) \]
\[ \text{Var}(G) = G''(1) + G'(1) - G''(1)^2, \quad (8.31) \]

when \( G \) is any function, since we frequently want to compute these combinations of derivatives.

The second-nicest thing about pgf’s is that they are comparatively simple functions of \( z \), in many important cases. For example, let’s look at the uniform distribution of order \( n \), in which the random variable takes on each of the values \( \{0, 1, \ldots, n-1\} \) with probability \( 1/n \). The pgf in this case is

\[ U_n(z) = \frac{1}{n} \sum_{i=0}^{n-1} z^i = \frac{1 - z^n}{n(1 - z)}, \quad \text{for } n \geq 1. \quad (8.32) \]

We have a closed form for \( U_n(z) \) because this is a geometric series.

But this closed form proves to be somewhat embarrassing: When we plug in \( z = 1 \) (the value of \( z \) that’s most critical for the pgf), we get the undefined ratio \( 0/0 \), even though \( U_n(z) \) is a polynomial that is perfectly well defined at any value of \( z \). The value \( U_n(1) = 1 \) is obvious from the non-closed form