variables. Indeed, if $X$ and $Y$ are random variables that take on nothing but integer values, the probability that $X + Y = n$ is
\[
\Pr(X + Y = n) = \sum_k \Pr(X = k \text{ and } Y = n - k).
\]
If $X$ and $Y$ are independent, we now have
\[
\Pr(X + Y = n) = \sum_k \Pr(X = k) \Pr(Y = n - k),
\]
a convolution. Therefore—and this is the punch line—
\[
G_{X+Y}(z) = G_X(z) G_Y(z), \quad \text{if } X \text{ and } Y \text{ are independent.} \tag{8.37}
\]
Earlier this chapter we observed that $\text{V}(X + Y) = \text{V}(X) + \text{V}(Y)$ when $X$ and $Y$ are independent. Let $F(z)$ and $G(z)$ be the pgf’s for $X$ and $Y$, and let $H(z)$ be the pgf for $X + Y$. Then
\[
H(z) = F(z)G(z),
\]
and our formulas (8.28) through (8.31) for mean and variance tell us that we must have
\[
\begin{align*}
\text{Mean}(H) &= \text{Mean}(F) + \text{Mean}(G); \tag{8.38} \\
\text{Var}(H) &= \text{Var}(F) + \text{Var}(G). \tag{8.39}
\end{align*}
\]
These formulas, which are properties of the derivatives $\text{Mean}(H) = H'(1)$ and $\text{Var}(H) = H''(1) + H'(1)^2$, aren’t valid for arbitrary function products $H(z) = F(z)G(z)$; we have
\[
\begin{align*}
H'(z) &= F'(z)G(z) + F(z)G'(z), \\
H''(z) &= F''(z)G(z) + 2F'(z)G'(z) + F(z)G''(z).
\end{align*}
\]
But if we set $z = 1$, we can see that (8.38) and (8.39) will be valid in general provided only that
\[
F(1) = G(1) = 1 \tag{8.40}
\]
and that the derivatives exist. The “probabilities” don’t have to be in $[0, 1]$ for these formulas to hold. We can normalize the functions $F(z)$ and $G(z)$ by dividing through by $F(1)$ and $G(1)$ in order to make this condition valid, whenever $F(1)$ and $G(1)$ are nonzero.

Mean and variance aren’t the whole story. They are merely two of an infinite series of so-called cumulant statistics introduced by the Danish astronomer Thorvald Nicolai Thiele [288] in 1903. The first two cumulants