\(\kappa_1\) and \(\kappa_2\) of a random variable are what we have called the mean and the variance; there also are higher-order cumulants that express more subtle properties of a distribution. The general formula

\[
\ln \mathcal{G}(e^t) = \frac{\kappa_1}{1!} t + \frac{\kappa_2}{2!} t^2 + \frac{\kappa_3}{3!} t^3 + \frac{\kappa_4}{4!} t^4 + \cdots \tag{8.41}
\]
defines the cumulants of all orders, when \(\mathcal{G}(z)\) is the pgf of a random variable.

Let's look at cumulants more closely. If \(\mathcal{G}(z)\) is the pgf for \(X\), we have

\[
\mathcal{G}(e^t) = \sum_{k \geq 0} \Pr(X = k) e^{kt} = \sum_{k, m \geq 0} \Pr(X = k) \frac{k^m t^m}{m!}
\]

\[
= 1 + \frac{\mu_1}{1!} t + \frac{\mu_2}{2!} t^2 + \frac{\mu_3}{3!} t^3 + \cdots \tag{8.42}
\]

where

\[
P_m = \sum_{k \geq 0} k^m \Pr(X = k) = E(X^m) \tag{8.43}
\]

This quantity \(P_m\) is called the “mth moment” of \(X\). We can take exponentials on both sides of (8.41), obtaining another formula for \(\mathcal{G}(e^t)\):

\[
\mathcal{G}(e^t) = 1 + \left( \frac{\kappa_1 t + \frac{1}{2} \kappa_2 t^2 + \cdots }{1!} \right) \cdot \left( \frac{\kappa_1 t + \frac{1}{2} \kappa_2 t^2 + \cdots }{2!} \right) \cdot \cdots
\]

\[
\mathcal{G}(e^t) = 1 + \frac{\kappa_1 t + \frac{1}{2} (\kappa_2 + \kappa_1^2) t^2 + \cdots }{2!}
\]

Equating coefficients of powers of \(t\) leads to a series of formulas

\[
\kappa_1 = \mu_1, \tag{8.44}
\]
\[
\kappa_2 = \mu_2 - \mu_1^2, \tag{8.45}
\]
\[
\kappa_3 = \mu_3 - 3\mu_1 \mu_2 + 2\mu_1^3, \tag{8.46}
\]
\[
\kappa_4 = \mu_4 - 4\mu_1 \mu_3 + 12\mu_1^2 \mu_2 - 3\mu_2^2 - 6\mu_1^4, \tag{8.47}
\]
\[
\kappa_5 = \mu_5 - 5\mu_1 \mu_4 + 20\mu_1^2 \mu_3 - 10\mu_2 \mu_3 + 30\mu_1^2 \mu_2^2 - 60\mu_1^3 \mu_2 + 24\mu_1^5, \tag{8.48}
\]

defining the cumulants in terms of the moments. Notice that \(\kappa_2\) is indeed the variance, \(E(X^2) - (EX)^2\), as claimed.

Equation (8.41) makes it clear that the cumulants defined by the product \(\mathcal{F}(z) \mathcal{G}(z)\) of two pgf’s will be the sums of the corresponding cumulants of \(\mathcal{F}(z)\) and \(\mathcal{G}(z)\), because logarithms of products are sums. Therefore all cumulants of the sum of independent random variables are additive, just as the mean and variance are. This property makes cumulants more important than moments.

"For these higher half-invariants we shall propose no special names." — T. N. Thiele [288]