A peculiar set of tennis players.

"A fast arithmetic computation shows that the sherry is always at least three years old. Taking computation further gives the vertigo."
- Revue du vin de France (Nov 1984)

is second best, \ldots, x_2^n is worst. When x_i plays x_k and j < k, the winner is x_j with probability \( p \) and x_k with probability \( 1 - p \), independent of the other matches. We assume that the same probability \( p \) applies to all j and k.

a. What’s the probability that x_1 wins the tournament?

b. What’s the probability that the nth round (the final match) is between the top two players, x_1 and x_2?

c. What’s the probability that the best \( 2^k \) players are the competitors in the kth-to-last round? (The previous questions were the cases k=0 and k= 1.)

d. Let N(n) be the number of essentially different tournament results; two tournaments are essentially the same if the matches take place between the same players and have the same winners. Prove that \( N(n) = 2^n \).

e. What’s the probability that x_2 wins the tournament?

f. Prove that if \( \frac{1}{2} < p < 1 \), the probability that x_j wins is strictly greater than the probability that x_{j+1} wins, for 1 \leq j < 2^n.

45 True sherry is made in Spain according to a multistage system called “Solera!” For simplicity we’ll assume that the winemaker has only three barrels, called A, B, and C. Every year a third of the wine from barrel C is bottled and replaced by wine from B; then B is topped off with a third of the wine from A; finally A is topped off with new wine. Let \( A(z), B(z), C(z) \) be probability generating functions, where the coefficient of \( z^n \) is the fraction of n-year-old wine in the corresponding barrel just after the transfers have been made.

a. Assume that the operation has been going on since time immemorial, so that we have a steady state in which \( A(z), B(z), \) and \( C(z) \) are the same at the beginning of each year. Find closed forms for these generating functions.

b. Find the mean and standard deviation of the age of the wine in each barrel, under the same assumptions. What is the average age of the sherry when it is bottled? How much of it is exactly 25 years old?

c. Now take the finiteness of time into account: Suppose that all three barrels contained new wine at the beginning of year 0. What is the average age of the sherry that is bottled at the beginning of year n?

46 Stefan Banach used to carry two boxes of matches, each containing n matches initially. Whenever he needed a light he chose a box at random, each with probability \( \frac{1}{2} \), independent of his previous choices. After taking out a match he’d put the box back in its pocket (even if the box became empty—all famous mathematicians used to do this). When his chosen box was empty he’d throw it away and reach for the other box.