They have two frisbees, initially at adjacent vertices as shown. In each
time interval, each frisbee is thrown either to the left or to the right
(along an edge of the pentagon) with equal probability. This process
continues until one person is the target of two frisbees simultaneously;
then the game stops. (All throws are independent of past history.)

a. Find the mean and variance of the number of pairs of throws.
b. Find a closed form for the probability that the game lasts more than
   100 steps, in terms of Fibonacci numbers.

49 Luke Snowwalker spends winter vacations at his mountain cabin. The
front porch has m pairs of boots and the back porch has n pairs. Every
time he goes for a walk he flips a (fair) coin to decide whether to leave
from the front porch or the back, and he puts on a pair of boots at that
porch and heads off. There’s a 50/50 chance that he returns to each
porch, independent of his starting point, and he leaves the boots at the
porch he returns to. Thus after one walk there will be m + [-1 , 0, or +1]
pairs on the front porch and n + [0, 1, or -1] pairs on the back porch.
If all the boots pile up on one porch and if he decides to leave from
the other, he goes without boots and gets frostbite, ending his vacation.
Assuming that he continues his walks until the bitter end, let \( P_N(m, n) \) be
the probability that he completes exactly N nonfrostbitten trips, starting
with m pairs on the front porch and n on the back. Thus, if both m
and n are positive,

\[
P_N(m, n) = \frac{1}{4} P_{N-1}(m - 1, n + 1) + \frac{1}{2} P_{N-1}(m, n) \\
+ \frac{1}{4} P_{N-1}(m + 1, n - 1);
\]

this follows because this first trip is either front/back, front/front, back/
back, or back/front, each with probability \( \frac{1}{4} \), and \( N - 1 \) trips remain.

a. Complete the recurrence for \( P_N(m, n) \) by finding formulas that hold
when m = 0 or n = 0. Use the recurrence to obtain equations that
hold among the probability generating functions

\[
g_{m,n}(z) = \sum_{N\geq0} P_N(m, n)z^N.
\]

b. Differentiate your equations and set \( z = 1 \), thereby obtaining relations
among the quantities \( g'_{m,n}(1) \). Solve these equations, thereby
determining the mean number of trips before frostbite.

c. Show that \( g_{m,n} \) has a closed form if we substitute \( z = 1 / \cos^2 \theta; \)

\[
g_{m,n}(\frac{1}{\cos^2 \theta}) = \frac{\sin(2m + 1)\theta + \sin(2n + 1)\theta}{\sin(2m + 2n + 2)} \cos \theta
\]