Consider the function
\[ H(z) = 1 + \frac{1 - z}{2} (z - 3 + \sqrt{1 - z}(9 - z)). \]

The purpose of this problem is to prove that \( H(z) = \sum_{k \geq 0} h_k z^k \) is a probability generating function, and to obtain some basic facts about it.

a. Let \((1 - z)^{3/2}/(1 - z)^{1/2} = \sum_{k \geq 0} c_k z^k.\) Prove that \(c_0 = 3,\) \(c_1 = -14/3,\)
\(c_2 = 37/27,\) and \(c_{k+1} = \frac{1}{k+1} \left(\frac{1}{2}\right)^{1/2} \left(\frac{8}{9}\right)^{k+3} \) for all \(k \geq 0.\) Hint: Use the identity
\[(9 - z)^{1/2} = 3(1 - z)^{1/2} (1 + \frac{8}{9} z/(1 - z))^{1/2}\]
and expand the last factor in powers of \(z/(1 - z).\)

b. Use part (a) and exercise 5.81 to show that the coefficients of \(H(z)\) are all positive.

c. Prove the amazing identity
\[ \sqrt{\frac{9 - H(z)}{1 - H(z)}} = \sqrt{\frac{9 - z}{1 - z}} + 2.\]

d. What are the mean and variance of \(H?\)

51 The state lottery in El Dorado uses the payoff distribution \(H\) defined in the previous problem. Each lottery ticket costs 1 doubloon, and the payoff is \(k\) doubloons with probability \(h_k.\) Your chance of winning with each ticket is completely independent of your chance with other tickets; in other words, winning or losing with one ticket does not affect your probability of winning with any other ticket you might have purchased in the same lottery.

a. Suppose you start with one doubloon and play this game. If you win \(k\) doubloons, you buy \(k\) tickets in the second game; then you take the total winnings in the second game and apply all of them to the third; and so on. If none of your tickets is a winner, you’re broke and you have to stop gambling. Prove that the pgf of your current holdings after \(n\) rounds of such play is
\[ 1 - \frac{4}{\sqrt{(9 - z)/(1 - z)} + 2n - 1} + \frac{4}{\sqrt{(9 - z)/(1 - z)} + 2n + 1}. \]

b. Let \(g_n\) be the probability that you lose all your money for the first time on the \(n\)th game, and let \(G(z) = g_1 z + g_2 z^2 + \cdots.\) Prove that \(G(1) = 1.\) (This means that you’re bound to lose sooner or later, with probability 1, although you might have fun playing in the meantime.) What are the mean and the variance of \(G?\)