EXACT ANSWERS are great when we can find them; there’s something very satisfying about complete knowledge. But there’s also a time when approximations are in order. If we run into a sum or a recurrence whose solution doesn’t have a closed form (as far as we can tell), we still would like to know something about the answer; we don’t have to insist on all or nothing. And even if we do have a closed form, our knowledge might be imperfect, since we might not know how to compare it with other closed forms.

For example, there is (apparently) no closed form for the sum

$$S_n = \sum_{k=0}^{n} \binom{3n}{k}.$$  

But it is nice to know that

$$S_n \sim 2 \binom{3n}{n}, \quad \text{as} \ n \to \infty;$$

we say that the sum is “asymptotic to” $2 \binom{3n}{n}$. It’s even nicer to have more detailed information, like

$$S_n = \binom{3n}{n} \left( 2 - \frac{4}{n} + O\left( \frac{1}{n^2} \right) \right), \quad \text{(9.1)}$$

which gives us a “relative error of order $1/n^2$.” But even this isn’t enough to tell us how big $S_n$ is, compared with other quantities. Which is larger, $S_n$ or the Fibonacci number $F_{4n}$? Answer: We have $S_2 = 22 > F_8 = 21$ when $n = 2$; but $F_{4n}$ is eventually larger, because $F_{4n} \sim \phi^{4n}/\sqrt{5}$ and $\phi^4 \approx 6.8541$, while

$$S_n = \sqrt{\frac{3}{\pi n}} \left( 6.75 \right)^n \left( 1 + \frac{151}{72n} + O\left( \frac{1}{n^2} \right) \right). \quad \text{(9.2)}$$

Our goal in this chapter is to learn how to understand and to derive results like this without great pain.