The word asymptotic stems from a Greek root meaning “not falling together!” When ancient Greek mathematicians studied conic sections, they considered hyperbolas like the graph of $y = \sqrt{1 + x^2}$, which has the lines $y = \chi$ and $y = -\chi$ as “asymptotes!” The curve approaches but never quite touches these asymptotes, when $x \to \infty$. Nowadays we use “asymptotic” in a broader sense to mean any approximate value that gets closer and closer to the truth, when some parameter approaches a limiting value. For us, asymptotics means “almost falling together!”

Some asymptotic formulas are very difficult to derive, well beyond the scope of this book. We will content ourselves with an introduction to the subject; we hope to acquire a suitable foundation on which further techniques can be built. We will be particularly interested in understanding the definitions of ‘~’ and ‘0’ and similar symbols, and we’ll study basic ways to manipulate asymptotic quantities.

### 9.1 A HIERARCHY

Functions of $n$ that occur in practice usually have different “asymptotic growth ratios”; one of them will approach infinity faster than another. We formalize this by saying that

$$f(n) \prec g(n) \iff \lim_{n \to \infty} \frac{f(n)}{g(n)} = 0.$$  \hspace{1cm} (9.3)

This relation is transitive: If $f(n) \prec g(n)$ and $g(n) \prec h(n)$ then $f(n) \prec h(n)$. We also may write $g(n) \succ f(n)$ if $f(n) \prec g(n)$. This notation was introduced in 1871 by Paul du Bois-Reymond [29].

For example, $n \prec n^2$; informally we say that $n$ grows more slowly than $n^2$. In fact,

$$n^\alpha \prec n^\beta \iff \alpha < \beta,$$  \hspace{1cm} (9.4)

when $\alpha$ and $\beta$ are arbitrary real numbers.

There are, of course, many functions of $n$ besides powers of $n$. We can use the $\prec$ relation to rank lots of functions into an asymptotic pecking order.