that includes entries like this:

\[ 1 < \log \log n < \log n < n^\varepsilon < n^{\varepsilon} < n^{\log n} < c^n < n^n < c^n. \]

(Here \( \varepsilon \) and \( c \) are arbitrary constants with \( 0 < \varepsilon < 1 < c \).)

All functions listed here, except 1, go to infinity as \( n \) goes to infinity. Thus when we try to place a new function in this hierarchy, we're not trying to determine whether it becomes infinite but rather how fast.

It helps to cultivate an expansive attitude when we're doing asymptotic analysis: We should think big, when imagining a variable that approaches infinity. For example, the hierarchy says that \( \log n < n^{0.0001} \), this might seem wrong if we limit our horizons to teeny-tiny numbers like one googol, \( n = 10^{100} \). For in that case, \( \log n = \log 10^{100} = 100 \), while \( n^{0.0001} = 100^{0.01} \approx 1.0233 \). But if we go up to a googolplex, \( n = 10^{10^{100}} \), then \( \log n = \log 10^{10^{100}} = 10^{10^{100}} \) pales in comparison with \( n^{0.0001} = 10^{10^{0.01}} \).

Even if \( \varepsilon \) is extremely small (smaller than, say, \( 1/10^{10^{100}} \)), the value of \( \log n \) will be much smaller than the value of \( n^\varepsilon \) if \( n \) is large enough. For if we set \( n = 10^{10^{10^k}} \), where \( k \) is so large that \( \varepsilon \geq 10^{-k} \), we have \( \log n = 10^{10^k} \) but \( n^\varepsilon \geq 10^{10^{10^k}} \). The ratio \( \log n/n^\varepsilon \) therefore approaches zero as \( n \to \infty \).

The hierarchy shown above deals with functions that go to infinity. Often, however, we're interested in functions that go to zero, so it's useful to have a similar hierarchy for those functions. We get one by taking reciprocals, because when \( f(n) \) and \( g(n) \) are never zero we have

\[ f(n) \prec g(n) \iff \frac{1}{g(n)} > \frac{1}{f(n)}. \]  

Thus, for example, the following functions (except 1) all go to zero:

\[ \frac{1}{c^n} < \frac{1}{n^n} < \frac{1}{n^{\log n}} < \frac{1}{n^\varepsilon} < \frac{1}{n^\varepsilon} < \frac{1}{\log n} < \frac{1}{\log \log n} < 1. \]

Let's look at a few other functions to see where they fit in. The number \( \pi(n) \) of primes less than or equal to \( n \) is known to be approximately \( n/\ln n \). Since \( 1/n^\varepsilon \prec 1/\ln n \sim 1 \), multiplying by \( n \) tells us that

\[ n^{1-\varepsilon} \prec \pi(n) \prec n. \]

We can in fact generalize (9.4) by noticing, for example, that

\[ n^{\alpha_1}(\log n)^{\beta_2}(\log \log n)^{\alpha_3} \prec n^{\beta_1}(\log n)^{\beta_2}(\log \log n)^{\beta_3}. \]

\[ \iff (\alpha_1, \alpha_2, \alpha_3) < (\beta_1, \beta_2, \beta_3). \]  

(9.6)

Here \((\alpha_1, \alpha_2, \alpha_3) < (\beta_1, \beta_2, \beta_3)\) means lexicographic order (dictionary order); in other words, either \( a_i < \beta_i \) or \( \alpha_i = \beta_i \) and \( \alpha_2 < \beta_2 \), or \( \alpha_i = \beta_i \) and \( \alpha_2 = \beta_2 \) and \( \alpha_3 < \beta_3 \).