Hardy’s main theorem about logarithmico-exponential functions is that they form an asymptotic hierarchy: If \( f(n) \) and \( g(n) \) are any functions in \( \mathbb{C} \), then either \( f(n) \sim g(n) \), or \( f(n) \gg g(n) \), or \( f(n) \ll g(n) \). In the last case there is, in fact, a constant \( \alpha \) such that

\[
\frac{f(n)}{g(n)} \sim \alpha.
\]

The proof of Hardy’s theorem is beyond the scope of this book; but it’s nice to know that the theorem exists, because almost every function we ever need to deal with is in \( \mathbb{C} \). In practice, we can generally fit a given function into a given hierarchy without great difficulty.

### 9.2 \( \mathcal{O} \) NOTATION

A wonderful notational convention for asymptotic analysis was introduced by Paul Bachmann in 1894 and popularized in subsequent years by Edmund Landau and others. We have seen it in formulas like

\[
H_n = \ln n + \gamma + \mathcal{O}\left(\frac{1}{n}\right),
\]

which tells us that the nth harmonic number is equal to the natural logarithm of \( n \) plus Euler’s constant, plus a quantity that is “Big Oh of 1 over \( n \)” This last quantity isn’t specified exactly; but whatever it is, the notation claims that its absolute value is no more than a constant times \( 1/n \).

The beauty of \( \mathcal{O} \)-notation is that it suppresses unimportant detail and lets us concentrate on salient features: The quantity \( \mathcal{O}(1/n) \) is negligibly small, if constant multiples of \( 1/n \) are unimportant.

Furthermore we get to use \( \mathcal{O} \) right in the middle of a formula. If we want to express (9.10) in terms of the notations in Section 9.1, we must transpose ‘\( \ln n + \gamma \)’ to the left side and specify a weaker result like

\[
H_n - \ln n - \gamma \ll \frac{\log \log n}{n}
\]

or a stronger result like

\[
H_n - \ln n - \gamma \ll \frac{1}{n}.
\]

The Big Oh notation allows us to specify an appropriate amount of detail in place, without transposition.

The idea of imprecisely specified quantities can be made clearer if we consider some additional examples. We occasionally use the notation ‘\( \pm 1 \)’ to stand for something that is either \( +1 \) or \( -1 \); we don’t know (or perhaps we don’t care) which it is, yet we can manipulate it in formulas.