it may well be a considerable overestimate of the actual local truncation error in some parts of the interval \([a, b]\). One use of Eq. (22) is to choose a step size that will result in a local truncation error no greater than some given tolerance level. For example, if the local truncation error must be no greater than \(\epsilon\), then from Eq. (22) we have

\[
h \leq \sqrt{2\epsilon/M}.
\]  

(23)

The primary difficulty in using any of Eqs. (21), (22), or (23) lies in estimating \(|\phi''(t)|\) or \(M\). However, the central fact expressed by these equations is that the local truncation error is proportional to \(h^2\). Thus, if \(h\) is reduced by a factor of \(\frac{1}{2}\), then the error is reduced by \(\frac{1}{4}\), and so forth.

More important than the local truncation error is the global truncation error \(E_n\). The analysis for estimating \(E_n\) is more difficult than that for \(e_n\). However, knowing the local truncation error, we can make an intuitive estimate of the global truncation error at a fixed \(\tilde{t} > t_0\) as follows. Suppose that we take \(n\) steps in going from \(t_0\) to \(\tilde{t} = t_0 + nh\). In each step the error is at most \(Mh^2/2\); thus the error in \(n\) steps is at most \(nMh^2/2\). Noting that \(n = (\tilde{t} - t_0)/h\), we find that the global truncation error for the Euler method in going from \(t_0\) to \(\tilde{t}\) is bounded by

\[
\frac{Mh^2}{2} = (\tilde{t} - t_0) \frac{Mh}{2}. \tag{24}
\]

This argument is not complete since it does not take into account the effect that an error at one step will have in succeeding steps. Nevertheless, it can be shown that the global truncation error in using the Euler method on a finite interval is no greater than a constant times \(h\); see Problem 23 for more details. The Euler method is called a first order method because its global truncation error is proportional to the first power of the step size.

Because it is more accessible, we will hereafter use the local truncation error as our principal measure of the accuracy of a numerical method, and for comparing different methods. If we have a priori information about the solution of the given initial value problem, we can use the result (21) to obtain more precise information about how the local truncation error varies with \(t\). As an example, consider the illustrative problem

\[
y' = 1 - t + 4y, \quad y(0) = 1 \tag{25}
\]
on the interval \(0 \leq t \leq 2\). Let \(y = \phi(t)\) be the solution of the initial value problem (25). Then, as noted previously,

\[
\phi(t) = (4t - 3 + 19e^{4t})/16
\]

and therefore

\[
\phi''(t) = 19e^{4t}.
\]

Equation (21) then states that

\[
e_{n+1} = \frac{19e^{4t_n}h^2}{2}, \quad t_n < \tilde{t}_n < t_n + h. \tag{26}
\]

The appearance of the factor 19 and the rapid growth of \(e^{4t}\) explain why the results in this section with \(h = 0.05\) were not very accurate.