N. G. de Bruijn begins his book Asymptotic Methods in Analysis by considering a Big E notation that helps us understand Big Oh. If we write $L(5)$ for a number whose absolute value is less than 5 (but we don’t say what the number is), then we can perform certain calculations without knowing the full truth. For example, we can deduce formulas such as $1 + L(5) = L(6)$; $L(2) + L(3) = L(5)$; $L(2)L(3) = L(6)$; $e^{L(5)} = L(e^5)$; and so on. But we cannot conclude that $L(5) - L(3) = L(2)$, since the left side might be $4 - 0$. In fact, the most we can say is $L(5) - L(3) = L(8)$.

Bachmann’s O-notation is similar to L-notation but it’s even less precise: $O(\alpha)$ stands for a number whose absolute value is at most a constant times $|\alpha|$. We don’t say what the number is and we don’t even say what the constant is. Of course the notion of a “constant” is nonsense if there is nothing variable in the picture, so we use O-notation only in contexts when there’s at least one quantity (say $n$) whose value is varying. The formula

$$f(n) = O(g(n))$$

for all $n$ means in this context that there is a constant $C$ such that

$$|f(n)| \leq C|g(n)|$$

for all $n$; (9.12)

and when $O(g(n))$ stands in the middle of a formula it represents a function $f(n)$ that satisfies (9.12). The values of $f(n)$ are unknown, but we do know that they aren’t too large. Similarly, de Bruijn’s ‘$L(n)$’ represents an unspecified function $f(n)$ whose values satisfy $|f(n)| < |n|$. The main difference between $L$ and $0$ is that O-notation involves an unspecified constant $C$; each appearance of $0$ might involve a different $C$, but each $C$ is independent of $n$.

For example, we know that the sum of the first $n$ squares is

$$\Delta_n = \frac{1}{3}n(n+1)(n+2) = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n.$$  

We can write

$$\Delta_n = O(n^3)$$

because $|\frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n| \leq \frac{1}{3}|n^3| + \frac{1}{2}|n^2| + \frac{1}{6}|n| = \frac{1}{3}|n^3| + \frac{1}{2}|n^2| + \frac{1}{6}|n| = |n^3|$ for all integers $n$. Similarly, we have the more specific formula

$$\Delta_n = \frac{1}{3}n^3 + O(n^2);$$

we can also be sloppy and throw away information, saying that

$$\Delta_n = O(n^{10}).$$

Nothing in the definition of $0$ requires us to give a best possible bound.