In each of Problems 7 through 12 find approximate values of the solution of the given initial value problem at \( t = 0.5, 1.0, 1.5, \) and 2.0.

(a) Use the Euler method with \( h = 0.025.\)
(b) Use the Euler method with \( h = 0.0125.\)
(c) Use the backward Euler method with \( h = 0.025.\)
(d) Use the backward Euler method with \( h = 0.0125.\)

\[
\begin{align*}
7. & \quad y' = 0.5 - t + 2y, \quad y(0) = 1 & 8. & \quad y' = 5t - 3\sqrt{y}, \quad y(0) = 2 \\
9. & \quad y' = \sqrt{t + y}, \quad y(0) = 3 & 10. & \quad y' = 2t + e^{-ty}, \quad y(0) = 1 \\
11. & \quad y' = (4 - ty)/(1 + y^2), \quad y(0) = -2 & 12. & \quad y' = (y^2 + 2ty)/(3 + t^2), \quad y(0) = 0.5 \\
13. & \quad \text{Complete the calculations leading to the entries in columns three and four of Table 8.1.1.} \\
14. & \quad \text{Complete the calculations leading to the entries in columns three and four of Table 8.1.2.} \\
15. & \quad \text{Using three terms in the Taylor series given in Eq. (12) and taking } h = 0.1, \text{ determine approximate values of the solution of the illustrative example } y' = 1 - t + 4y, \ y(0) = 1 \text{ at } t = 0.1 \text{ and } 0.2. \text{ Compare the results with those using the Euler method and with the exact values.} \\
\text{Hint: If } y' = f(t, \ y), \text{ what is } y''? \\
\end{align*}
\]

In each of Problems 16 and 17 estimate the local truncation error for the Euler method in terms of the solution of the solution \( y = \phi(t). \) Obtain a bound for \( e_{n+1} \) in terms of \( t \) and \( \phi(t) \) that is valid on the interval \( 0 \leq t \leq 1. \) By using a formula for the solution obtain a more accurate error bound for \( e_{n+1}. \) For \( h = 0.1 \) compute a bound for \( e_i \) and compare it with the actual error at \( t = 0.1. \) Also compute a bound for the error \( e_i \) in the fourth step.

16. \( y' = 2y - 1, \quad y(0) = 1 \) 
17. \( y' = \frac{1}{2} - t + 2y, \quad y(0) = 1 \)

In each of Problems 18 through 21 obtain a formula for the local truncation error for the Euler method in terms of \( t \) and the solution \( \phi. \)

18. \( y' = t^2 + y^2, \quad y(0) = 1 \) 
19. \( y' = 5t - 3\sqrt{y}, \quad y(0) = 2 \) 
20. \( y' = \sqrt{t + y}, \quad y(1) = 3 \) 
21. \( y' = 2t + e^{-ty}, \quad y(0) = 1 \)

22. Consider the initial value problem
\[
y' = \cos 5\pi t, \quad y(0) = 1.
\]

(a) Determine the solution \( y = \phi(t) \) and draw a graph of \( y = \phi(t) \) for \( 0 \leq t \leq 1. \)
(b) Determine approximate values of \( \phi(t) \) at \( t = 0.2, 0.4, \) and 0.6 using the Euler method with \( h = 0.2. \) Draw a broken-line graph for the approximate solution and compare it with the graph of the exact solution.
(c) Repeat the computation of part (b) for \( 0 \leq t \leq 0.4, \) but take \( h = 0.1. \)
(d) Show by computing the local truncation error that neither of these step sizes is sufficiently small. Determine a value of \( h \) to ensure that the local truncation error is less than 0.05 throughout the interval \( 0 \leq t \leq 1. \) That such a small value of \( h \) is required results from the fact that \( \max |\phi''(t)| \) is large.

23. In this problem we discuss the global truncation error associated with the Euler method for the initial value problem \( y' = f(t, y), \ y(t_0) = y_0. \) Assuming that the functions \( f \) and \( f_y \) are continuous in a region \( R \) of the \( ty \)-plane that includes the point \( (t_0, \ y_0), \) it can be shown that there exists a constant \( L \) such that \( |f(t, y) - f(t, \ y) | < L|y - \ y|, \) where \( (t, \ y) \) and \( (t, \ y) \) are any two points in \( R \) with the same \( t \) coordinate (see Problem 15 of Section 2.8). Further, we assume that \( f_y \) is continuous, so the solution \( \phi \) has a continuous second derivative.