8.1 The Euler or Tangent Line Method

(a) Using Eq. (20) show that

\[ |E_{n+1}| \leq |E_n| + h|f(t_n, \phi(t_n)) - f(t_n, y_n)| + \frac{1}{2}h^2|\phi''(T_n)| \leq \alpha |E_n| + \beta h^2, \]  

(i)

where \( \alpha = 1 + hL \) and \( \beta = \max |\phi''(t)|/2 \) on \( t_0 \leq t \leq t_n \).

(b) Accepting without proof that if \( E_0 = 0 \), and if \( |E_n| \) satisfies Eq. (i), then \( |E_n| \leq \beta h^2 (\alpha^n - 1)/(\alpha - 1) \) for \( \alpha \neq 1 \), show that

\[ |E_n| \leq \beta h^2 \left( \frac{(1 + hL)^n - 1}{L} \right) \]  

(ii)

Equation (ii) gives a bound for \( |E_n| \) in terms of \( h, L, n, \) and \( \beta \). Notice that for a fixed \( h \), this error bound increases with increasing \( n \); that is, the error bound increases with distance from the starting point \( t_0 \).

(c) Show that \((1 + hL)^n \leq e^{nhL}\); hence

\[ |E_n| \leq \frac{e^{nhL} - 1}{L} \beta h = \frac{e^{(t_n-t_0)L} - 1}{L} \beta h. \]

For a fixed point \( \bar{t} = t_0 + nh \) [that is, \( nh \) is constant and \( h = (\bar{t} - t_0)/n \)] this error bound is of the form of a constant times \( h \) and approaches zero as \( h \to 0 \). Also note that for \( nhL = (\bar{t} - t_0)L \) small the right side of the preceding equation is approximately \( nh^2 \beta = (\bar{t} - t_0)\beta h \), which was obtained in Eq. (24) by an intuitive argument.

24. Derive an expression analogous to Eq. (21) for the local truncation error for the backward Euler formula.

Hint: Construct a suitable Taylor approximation to \( \phi(t) \) about \( t = t_{n+1} \).

25. Using a step size \( h = 0.05 \) and the Euler method, but retaining only three digits throughout the computations, determine approximate values of the solution at \( t = 0.1, 0.2, 0.3, \) and \( 0.4 \) for each of the following initial value problems.

(a) \( y' = 1 - t + 4y, \quad y(0) = 1 \)
(b) \( y' = 3 + t - y, \quad y(0) = 1 \)
(c) \( y' = 2y - 3t, \quad y(0) = 1 \)

Compare the results with those obtained in Example 1 and in Problems 1 and 3. The small differences between some of those results rounded to three digits and the present results are due to round-off error. The round-off error would become important if the computation required many steps.

26. The following problem illustrates a danger that occurs because of round-off error when nearly equal numbers are subtracted, and the difference then multiplied by a large number. Evaluate the quantity

\[
\begin{array}{ccc}
1000 & 6.010 & 18.04 \\
2.004 & 6.000 &
\end{array}
\]

as follows.

(a) First round each entry in the determinant to two digits.
(b) First round each entry in the determinant to three digits.
(c) Retain all four digits. Compare this value with the results in parts (a) and (b).

27. The distributive law \( a(b - c) = ab - ac \) does not hold, in general, if the products are rounded off to a smaller number of digits. To show this in a specific case take \( a = 0.22, b = 3.19, \) and \( c = 2.17 \). After each multiplication round off the last digit.