8.2 Improvements on the Euler Method

Difficulty can be overcome by replacing $y_{n+1}$ on the right side of Eq. (4) by the value obtained using the Euler formula (3). Thus

$$y_{n+1} = y_n + \frac{f(t_n, y_n) + f(t_n + h, y_n + hf(t_n, y_n))}{2} h$$

(5)

where $t_{n+1}$ has been replaced by $t_n + h$.

Equation (5) gives an explicit formula for computing $y_{n+1}$, the approximate value of $\phi(t_{n+1})$, in terms of the data at $t_n$. This formula is known as the improved Euler formula or the Heun formula. The improved Euler formula (5) does represent an improvement over the Euler formula (3) because the local truncation error in using Eq. (5) is proportional to $h^3$, while for the Euler method it is proportional to $h^2$. This error estimate for the improved Euler formula is established in Problem 14. It can also be shown that for a finite interval the global truncation error for the improved Euler formula is bounded by a constant times $h^2$, so this method is a second order method. Note that this greater accuracy is achieved at the expense of more computational work, since it is now necessary to evaluate $f(t, y)$ twice in order to go from $t_n$ to $t_{n+1}$.

If $f(t, y)$ depends only on $t$ and not on $y$, then solving the differential equation $y' = f(t, y)$ reduces to integrating $f(t)$. In this case the improved Euler formula (5) becomes

$$y_{n+1} - y_n = \frac{h}{2} [f(t_n) + f(t_n + h)],$$

(6)

which is just the trapezoid rule for numerical integration.

**Example 1**

Use the improved Euler formula (5) to calculate approximate values of the solution of the initial value problem

$$y' = 1 - t + 4y, \quad y(0) = 1.$$  

(7)

To make clear exactly what computations are required, we show a couple of steps in detail. For this problem $f(t, y) = 1 - t + 4y$; hence

$$f_n = 1 - t_n + 4y_n$$

and

$$f(t_n + h, y_n + hf_n) = 1 - (t_n + h) + 4(y_n + hf_n).$$

Further, $t_0 = 0$, $y_0 = 1$, and $f_0 = 1 - t_0 + 4y_0 = 5$. If $h = 0.025$, then

$$f(t_0 + h, y_0 + hf_0) = 1 - 0.025 + 4[1 + (0.025)(5)] = 5.475.$$  

Then, from Eq. (5),

$$y_1 = 1 + (0.5)(5 + 5.475)(0.025) = 1.1309375.$$  

(8)

At the second step we must calculate

$$f_1 = 1 - 0.025 + 4(1.1309375) = 5.49875,$$

$$y_1 + hf_1 = 1.1309375 + (0.025)(5.49875) = 1.26840625.$$