9.3 O MANIPULATION

Like any mathematical formalism, the O-notation has rules of manipulation that free us from the grungy details of its definition. Once we prove that the rules are correct, using the definition, we can henceforth work on a higher plane and forget about actually verifying that one set of functions is contained in another. We don’t even need to calculate the constants C that are implied by each O, as long as we follow rules that guarantee the existence of such constants.

For example, we can prove once and for all that

\[ n^m = O(n^{m'}) \], when \( m \leq m' \);
\[ O(f(n)) + O(g(n)) = O(|f(n)| + |g(n)|) \].

Then we can say immediately that \( \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n = O(n^3) + O(n^3) + O(n^3) = O(n^3) \), without the laborious calculations in the previous section.

Here are some more rules that follow easily from the definition:

\[ f(n) = O(f(n)) \];
\[ c \cdot O(f(n)) = O(f(n)) \], if \( c \) is constant;
\[ O(O(f(n))) = O(f(n)) \];
\[ O(f(n))O(g(n)) = O(f(n)g(n)) \];
\[ O(f(n)g(n)) = f(n)O(g(n)) \].

Exercise 9 proves (9.22), and the proofs of the others are similar. We can always replace something of the form on the left by what’s on the right, regardless of the side conditions on the variable \( n \).

Equations (9.27) and (9.23) allow us to derive the identity \( O(f(n))^2 = O((f(n))^2) \). This sometimes helps avoid parentheses, since we can write

\[ O(\log n)^2 \] instead of \( O(|\log n|^2) \).

Both of these are preferable to ‘\( O(\log^2 n) \)’, which is ambiguous because some authors use it to mean ‘\( O(\log \log n) \)’.

Can we also write

\[ O(\log n)^{-1} \] instead of \( O((\log n)^{-1}) \)?

No! This is an abuse of notation, since the set of functions \( l/O(\log n) \) is neither a subset nor a superset of \( 0 \ (1/\log n) \). We could legitimately substitute \( O((\log n)^{-1}) \) for \( O(\log n)^{-1} \), but this would be awkward. So we’ll restrict our use of “exponents outside the 0” to constant, positive integer exponents.

The secret of being a bore is to tell everything.

— Voltaire