(Here we assume that $n \to \infty$; similar formulas hold for $\ln(1 + 0(f(x)))$ and $e^{O(f(x))}$ as $x \to 0$.) For example, let $\ln(1 + g(n))$ be any function belonging to the left side of (9.36). Then there are constants $C, n_0, c$ such that

$$|g(n)| \leq C|f(n)| \leq c < 1,$$

for all $n \geq n_0$.

It follows that the infinite sum

$$\ln(1 + g(n)) \approx g(n) \cdot \left(1 + \frac{1}{2}g(n) + \frac{1}{3}g(n)^2 + \cdots \right)$$

converges for all $n \geq n_0$, and the parenthesized series is bounded by the constant $1 + \frac{1}{2}c + \frac{1}{3}c^2 + \cdots$. This proves (9.36), and the proof of (9.37) is similar. Equations (9.36) and (9.37) combine to give the useful formula

$$\left(1 + O(f(n))\right)^{O(g(n))} = 1 + O(f(n)g(n)), \quad \text{if } f(n) < 1 \text{ and } f(n)g(n) = O(1) \quad (9.38)$$

**Problem 1: Return to the Wheel of Fortune.**

Let’s try our luck now at a few asymptotic problems. In Chapter 3 we derived equation (3.13) for the number of winning positions in a certain game:

$$W = \left[\frac{N}{K}\right] + \frac{1}{2}K^2 + \frac{2}{3}K - 3, \quad K = \lfloor \sqrt{N} \rfloor.$$ 

And we promised that an asymptotic version of $W$ would be derived in Chapter 9. Well, here we are in Chapter 9; let’s try to estimate $W$, as $N \to \infty$.

The main idea here is to remove the floor brackets, replacing $K$ by $N^{1/3} + O(1)$. Then we can go further and write

$$K = N^{1/3}(1 + O(N^{-1/3})) ;$$

this is called “pulling out the large part!” (We will be using this trick a lot.) Now we have

$$K^2 = N^{2/3}(1 + O(N^{-1/3}))^2$$

$$= N^{2/3}(1 + O(N^{-1/3})) = N^{2/3} + O(N^{1/3})$$

by (9.38) and (9.26). Similarly

$$\left[\frac{N}{K}\right] = N^{1-1/3}(1 + O(N^{-1/3}))^{-1} + O(1)$$

$$= N^{2/3}(1 + O(N^{-1/3})) + O(1) = N^{2/3} + O(N^{1/3}).$$

It follows that the number of winning positions is

$$W = N^{2/3} + O(N^{1/3}) + \frac{1}{2}(N^{2/3} + O(N^{1/3})) + O(N^{1/3}) + O(1)$$

$$= \frac{3}{2}N^{2/3} + O(N^{1/3}). \quad (9.39)$$