8.3 The Runge-Kutta Method

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(c) Show that if \( f(t, y) \) is linear in \( t \) and \( y \), then \( e_{n+1} = \phi'''(t_n)h^3/6 \), where \( t_n < t_n < t_{n+1} \).

Hint: What are \( f_{1n}, f_{2n}, \) and \( f_{3n} \)?

15. Consider the improved Euler method for solving the illustrative initial value problem \( y' = 1 + t + 4y, \quad y(0) = 1 \). Using the result of Problem 14(c) and the exact solution of the initial value problem, determine \( e_{n+1} \) and a bound for the error at any step on \( 0 \leq t \leq 1 \). Compare this error with the one obtained in Eq. (26) of Section 8.1 using the Euler method. Also obtain a bound for \( e_1 \) for \( h = 0.1 \) and compare it with Eq. (27) of Section 8.1.

In each of Problems 16 and 17 use the actual solution \( \phi(t) \) to determine \( e_{n+1} \) and a bound for \( e_n \) at any step on \( 0 \leq t \leq 1 \) for the improved Euler method for the given initial value problem. Also obtain a bound for \( e_1 \) for \( h = 0.1 \) and compare it with the similar estimate for the Euler method and with the actual error using the improved Euler method.

16. \( y' = 2y - 1, \quad y(0) = 1 \)
17. \( y' = 0.5 - t + 2y, \quad y(0) = 1 \)

In each of Problems 18 through 21 carry out one step of the Euler method and of the improved Euler method using the step size \( h = 0.1 \). Suppose that a local truncation error no greater than 0.0025 is required. Estimate the step size that is needed for the Euler method to satisfy this requirement at the first step.

18. \( y' = 0.5 - t + 2y, \quad y(0) = 1 \)
19. \( y' = 5t - 3\sqrt{y}, \quad y(0) = 2 \)
20. \( y' = \sqrt{t + y}, \quad y(0) = 3 \)
21. \( y' = (y^2 + 2ty)/(3 + t^2), \quad y(0) = 0.5 \)

22. The modified Euler formula for the initial value problem \( y' = f(t, y), \quad y(t_0) = y_0 \) is given by

\[
y_{n+1} = y_n + hf(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hf(t_n, y_n)).
\]

Following the procedure outlined in Problem 14, show that the local truncation error in the modified Euler formula is proportional to \( h^3 \).

In each of Problems 23 through 26 use the modified Euler formula of Problem 22 with \( h = 0.05 \) to compute approximate values of the solution of the given initial value problem at \( t = 0.1, 0.2, 0.3, \) and 0.4. Compare the results with those obtained in Problems 1 through 4.

23. \( y' = 3 + t - y, \quad y(0) = 1 \)
24. \( y' = 5t - 3\sqrt{y}, \quad y(0) = 2 \)
25. \( y' = 2y - 3t, \quad y(0) = 1 \)
26. \( y' = 2t + e^{-t}, \quad y(0) = 1 \)

27. Show that the modified Euler formula of Problem 22 is identical to the improved Euler formula of Eq. (5) for \( y' = f(t, y) \) if \( f \) is linear in both \( t \) and \( y \).

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In preceding sections we have introduced the Euler formula, the backward Euler formula, and the improved Euler formula as ways to solve the initial value problem

\[
y' = f(t, y), \quad y(t_0) = y_0 \quad (1)
\]