Here we assume that \( n \to \infty \); similar formulas hold for \( \ln(1 + 0(f(x))) \) and \( e^{O(f(x))} \) as \( x \to 0 \). For example, let \( \ln(1 + g(n)) \) be any function belonging to the left side of (9.36). Then there are constants \( C, n_0, c \) such that

\[
|g(n)| \leq C |f(n)| \leq c < 1 \quad \text{for all } n \geq n_0.
\]

It follows that the infinite sum

\[
\ln(1 + g(n)) = g(n) \cdot \left(1 - \frac{1}{2} g(n) + \frac{1}{2} g(n)^2 - \cdots \right)
\]

converges for all \( n \geq n_0 \) and the parenthesized series is bounded by the constant \( 1 + \frac{1}{2} c + \frac{1}{2} c^2 + \cdots \). This proves (9.36), and the proof of (9.37) is similar. Equations (9.36) and (9.37) combine to give the useful formula

\[
\left(1 + O(f(n))\right)^{O(g(n))} = 1 + O(f(n)g(n)) \quad \text{if } f(n) < 1 \text{ and } f(n)g(n) = O(1). \tag{9.38}
\]

**Problem 1: Return to the Wheel of Fortune.**

Let’s try our luck now at a few asymptotic problems. In Chapter 3 we derived equation (3.13) for the number of winning positions in a certain game:

\[
W = \lceil N/K \rceil + \frac{1}{2} K^2 + \frac{2}{3} K - 3, \quad K = \sqrt[3]{N}.
\]

And we promised that an asymptotic version of \( W \) would be derived in Chapter 9. Well, here we are in Chapter 9; let’s try to estimate \( W \), as \( N \to \infty \).

The main idea here is to remove the floor brackets, replacing \( K \) by \( N^{1/3} + 0 \ (1) \). Then we can go further and write

\[
K = N^{1/3} (1 + O(N^{-1/3})),
\]

this is called “pulling out the large part!” (We will be using this trick a lot.) Now we have

\[
K^2 = N^{2/3} (1 + O(N^{-1/3}))^2 = N^{2/3} (1 + O(N^{-1/3})) = N^{2/3} + O(N^{1/3})
\]

by (9.38) and (9.26). Similarly

\[
\lfloor N/K \rfloor = N^{1 - 1/3} (1 + O(N^{-1/3}))^{-1} + O(1) = N^{2/3} (1 + O(N^{-1/3})) + O(1) = N^{2/3} + O(N^{1/3}).
\]

It follows that the number of winning positions is

\[
W = N^{2/3} + O(N^{1/3}) + \frac{1}{2} (N^{2/3} + O(N^{1/3})) + O(1)
\]

\[
= \frac{3}{2} N^{2/3} + O(N^{1/3}). \tag{9.39}
\]