Notice how the 0 terms absorb one another until only one remains; this is
typical, and it illustrates why $O$-notation is useful in the middle of a formula.

**Problem 2: Perturbation of Stirling’s formula.**

Stirling’s approximation for $n!$ is undoubtedly the most famous asymptotic formula of all. We will prove it later in this chapter; for now, let’s just try to get better acquainted with its properties. We can write one version of the approximation in the form

$$n! = \frac{n^n e^{-n}}{\sqrt{2\pi n}} \left(1 + \frac{a}{n} + \frac{b}{n^2} + O(n^{-3})\right), \quad \text{as } n \to \infty, \quad (9.40)$$

for certain constants $a$ and $b$. Since this holds for all large $n$, it must also be asymptotically true when $n$ is replaced by $n - 1$:

$$n! = \frac{n^n e^{-n}}{\sqrt{2\pi n}} \left(1 + \frac{a}{n} + \frac{b}{(n-1)^2} + O((n-1)^{-3})\right), \quad (9.41)$$

We know, of course, that $(n - 1)! = \frac{n!}{n}$; hence the right-hand side of this formula must simplify to the right-hand side of $(9.40)$, divided by $n$.

Let us therefore try to simplify $(9.41)$. The first factor becomes tractable if we pull out the large part:

$$\sqrt{2\pi n} = \sqrt{2\pi} \left(1 - n^{-1}\right)^{1/2} = \sqrt{2\pi} \left(1 + \frac{1}{2n} + O(n^{-2})\right)$$

Equation $(9.35)$ has been used here.

Similarly we have

$$\frac{a}{n} = \frac{a}{n}(1 - n^{-1})^{-1} = \frac{a}{n} + \frac{a}{n^2} + O(n^{-3});$$

$$\frac{b}{(n-1)^2} = \frac{b}{n^2} (1 - n^{-1})^{-2} = \frac{b}{n^2} + O(n^{-3});$$

$$O((n-1)^{-3}) = O(n^{-3}(1 - n^{-1})^{-3}) = O(n^{-3}).$$

The only thing in $(9.41)$ that’s slightly tricky to deal with is the factor $(n - 1)^{n-1}$, which equals

$$n^n (1 - n^{-1})^{n-1} = n^{n-1} (1 - n^{-1})^n (1 + n^{-1} + n^{-2} + O(n^{-3})).$$