Notice how the 0 terms absorb one another until only one remains; this is typical, and it illustrates why O-notation is useful in the middle of a formula.

**Problem 2: Perturbation of Stirling’s formula.**

Stirling’s approximation for \( n! \) is undoubtedly the most famous asymptotic formula of all. We will prove it later in this chapter; for now, let’s just try to get better acquainted with its properties. We can write one version of the approximation in the form

\[
n! = \sqrt{2\pi n^\frac{n}{e}} \left(1 + \frac{a}{n} + \frac{b}{n^2} + O(n^{-3})\right), \quad \text{as } n \to \infty, \quad (9.40)
\]

for certain constants \( a \) and \( b \). Since this holds for all large \( n \), it must also be asymptotically true when \( n \) is replaced by \( n - 1 \):

\[
(n-1)! = \sqrt{2\pi(n-1)^{(n-1)}}
\]

\[
\times \left(1 + \frac{a}{n-1} + \frac{b}{(n-1)^2} + O((n-1)^{-3})\right) \quad (9.41)
\]

We know, of course, that \((n - 1)! = n!/n\); hence the right-hand side of this formula must simplify to the right-hand side of \((9.40)\), divided by \( n \).

Let us therefore try to simplify \((9.41)\). The first factor becomes tractable if we pull out the large part:

\[
\sqrt{2\pi(n-1)} = \sqrt{2\pi n} \left(1 - n^{-1}\right)^{1/2}
\]

\[
= \sqrt{2\pi n} \left(1 - \frac{1}{2n} \frac{1}{8n^2} + O(n^{-3})\right)
\]

Equation \((9.35)\) has been used here.

Similarly we have

\[
\frac{a}{n-1} = \frac{a}{n} \left(1 - n^{-1}\right)^{-1} = \frac{a}{n} + \frac{a}{n^2} + O(n^{-3});
\]

\[
\frac{b}{(n-1)^2} = \frac{b}{n^2} \left(1 - n^{-1}\right)^{-2} = \frac{b}{n^2} + O(n^{-3});
\]

\[
O((n-1)^{-3}) = O(n^{-3}(1 - n^{-1})^{-3}) = O(n^{-3}).
\]

The only thing in \((9.41)\) that’s slightly tricky to deal with is the factor \((n - 1)^{n-1}\), which equals

\[
n^{n-1} (1 - n^{-1})^{n-1} = n^{n-1} (1 - n^{-1})^n (1 + n^{-1} + n^{-2} + O(n^{-3})).
\]