since the right side approaches zero as \( n \to \infty \). OK, we know that \( p = O(n^2) \);

therefore \( \log p = O(\log n) \) and \( \log \log p = O(\log \log n) \). We can now conclude from (9.45) that

\[
\ln p = \ln n + O(\log \log n);
\]

in fact, with this new estimate in hand we can conclude that \( \ln \ln p = \ln \ln n + O(\log \log n) \), and (9.45) now yields

\[
\ln p = \ln n + \ln \ln n + O(\log \log n / \log n).
\]

And we can plug this into the right-hand side of (9.44), obtaining

\[
p = n \ln n + n \ln \ln n + O(n)
\]

This is the approximate size of the \( n \)th prime.

We can refine this estimate by using a better approximation of \( n(n) \) in place of (9.42). The next term of (9.31) tells us that

\[
n = \frac{p}{\ln p} + \left( \frac{p}{\ln^2 p} \right) + O\left( \frac{p}{(\log p)^2} \right);
\]

proceeding as before, we obtain the recurrence

\[
p = n(n) \ln p \left( 1 + (\ln p) - \ln (\ln p) \right) + O(1/\log n)
\]

which has a relative error of \( O(1/\log n) \) instead of \( O(1/\log n) \). Taking logarithms and retaining proper accuracy (but not too much) now yields

\[
\ln p = \ln n + \ln \ln p + O(1/\log n)
\]

and

\[
\ln \ln p = \ln \ln n + \ln \ln \ln n + O(1/\log n).
\]

Finally we substitute these results into (9.47) and our answer finds its way out:

\[
P_n = n \ln n + n \ln \ln n - n + n \ln \ln n / \ln n + O\left( \frac{n}{\log n} \right).
\]

For example, when \( n = 10^6 \) this estimate comes to \( 15631363.8 + O(n/\log n) \); the millionth prime is actually 15485863. Exercise 21 shows that a still more accurate approximation to \( P_n \) results if we begin with a still more accurate approximation to \( \pi(n) \) in place of (9.46).