Problem 4: A sum from an old final exam.

When Concrete Mathematics was first taught at Stanford University during the 1970-1971 term, students were asked for the asymptotic value of the sum

\[ S_n = \sum_{k=1}^{\frac{n}{2}} \frac{1}{n^2 + k} \]

with an absolute error of \( O(n^{-7}) \). Let’s imagine that we’ve just been given this problem on a (take-home) final; what is our first instinctive reaction?

No, we don’t panic. Our first reaction is to think big. If we set \( n = 10^{100} \), say, and look at the sum, we see that it consists of \( n \) terms, each of which is slightly less than \( \frac{1}{n^2} \); hence the sum is slightly less than \( \frac{1}{n} \). In general, we can usually get a decent start on an asymptotic problem by taking stock of the situation and getting a ballpark estimate of the answer.

Let’s try to improve the rough estimate by pulling out the largest part of each term. We have

\[ \frac{1}{n^2 + k} = \frac{1}{n^2(1 + k/n^2)} = \frac{1}{n^2} \left( 1 - \frac{k}{n^2} + \frac{k^2}{n^4} - \frac{k^3}{n^6} + O\left(\frac{k^4}{n^8}\right) \right), \]

and so it’s natural to try summing all these approximations:

\[ \frac{1}{n^2 + 1} = \frac{1}{n^2} - \frac{1}{n^4} + \frac{1}{n^6} - \frac{1}{n^8} + O\left(\frac{1}{n^{10}}\right) \]

\[ \frac{1}{n^2 + 2} = \frac{1}{n^2} - \frac{2}{n^4} + \frac{2}{n^6} - \frac{2}{n^8} + O\left(\frac{2}{n^{10}}\right) \]

\[ \frac{1}{n^2 + n} = \frac{1}{n^2} - \frac{n}{n^4} + \frac{n^2}{n^6} - \frac{n^3}{n^8} + O\left(\frac{n^4}{n^{10}}\right) \]

\[ S_n = \frac{n}{n^2} - \frac{n(n+1)}{2n^4} + \cdots. \]

It looks as if we’re getting \( S_n = n^{-1} - \frac{1}{2} n^{-2} + O(n^{-3}) \), based on the sums of the first two columns; but the calculations are getting hairy.

If we persevere in this approach, we will ultimately reach the goal; but we won’t bother to sum the other columns, for two reasons: First, the last column is going to give us terms that are \( O(n^{-6}) \), when \( n/2 \leq k \leq n \), so we will have an error of \( O(n^{-5}) \); that’s too big, and we will have to include yet another column in the expansion. Could the exam-giver have been so sadistic? We suspect that there must be a better way. Second, there is indeed a much better way, staring us right in the face.

Do pajamas have buttons?