Namely, we know a closed form for $S_1$: It’s just $H_{n^2+n} = H_{n^2}$. And we know a good approximation for harmonic numbers, so we just apply it twice:

$$H_{n^2+n} = \ln(n^2+n) + \gamma + \frac{\gamma}{n} - \frac{1}{2n^2} + O\left(\frac{1}{n^3}\right);$$
$$H_{n^2} = \ln n^2 + \gamma + \frac{1}{2n^2} + O\left(\frac{1}{n^3}\right).$$

Now we can pull out large terms and simplify, as we did when looking at Stirling’s approximation. We have

$$\ln(n^2+n) = \ln n^2 + 1 + \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \cdots;$$
$$\frac{1}{n^2+n} = \frac{1}{n^2} + \frac{1}{n^3} + \frac{1}{n^4} - \cdots;$$
$$\frac{1}{(n^2+n)^2} = \frac{1}{n^4} + \frac{3}{n^5} + \frac{3}{n^6} - \cdots.$$

So there’s lots of helpful cancellation, and we find

$$S_n = n^{-1} - \frac{1}{2}n^{-2} + \frac{1}{3}n^{-3} - \frac{1}{4}n^{-4} + \frac{1}{5}n^{-5} - \frac{1}{6}n^{-6}$$
$$- \frac{1}{7}n^{-7} + \frac{1}{8}n^{-8} - \frac{1}{9}n^{-9} + \frac{1}{10}n^{-10}$$
$$+ \frac{1}{11}n^{-11} - \frac{1}{12}n^{-12} + O(n^{-7}).$$

(9.50)

It would be nice if we could check this answer numerically, as we did when we derived exact results in earlier chapters. Asymptotic formulas are harder to verify; an arbitrarily large constant may be hiding in a $O$ term, so any numerical test is inconclusive. But in practice, we have no reason to believe that an adversary is trying to trap us, so we can assume that the unknown $O$-constants are reasonably small. With a pocket calculator we find that $S_4 = \frac{1}{17} + \frac{1}{18} + \frac{1}{19} + \frac{1}{20} = 0.2170107$; and our asymptotic estimate when $n = 4$ comes to

$$\frac{1}{4}\left(1 + \frac{1}{4}\left(-\frac{1}{2} + \frac{1}{4}\left(-\frac{1}{6} + \frac{1}{4}\left(\frac{1}{4} + \frac{1}{4}\left(-\frac{2}{3} + \frac{1}{4}\left(-\frac{1}{12}\right)\right)\right)\right)\right) = 0.2170125.$$

If we had made an error of, say, $\frac{1}{12}$ in the term for $n^{-6}$, a difference of $\frac{1}{4096}$ would have shown up in the fifth decimal place; so our asymptotic answer is probably correct.