Namely, we know a closed form for $S$: It’s just $H_{n^2+n} = H_{n^2}$. And we know a good approximation for harmonic numbers, so we just apply it twice:

$$
H_{n^2+n} = \ln(n^2 + n) + \gamma + \frac{1}{2(n^2 + n)} \left( \frac{1}{12(n^2 + n)} \right)^2 + O\left( \frac{1}{n^8} \right);
$$

$$
H_{n^2} = \ln n^2 + \gamma + \frac{1}{2n^2} - \frac{1}{12n^4} + O\left( \frac{1}{n^8} \right).
$$

Now we can pull out large terms and simplify, as we did when looking at Stirling’s approximation. We have

$$
\ln(n^2 + n) = \ln n^2 + 1n \left( 1 + \frac{1}{n} \right) = \ln n^2 + \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \cdots;
$$

$$
\frac{1}{n^2 + n} = \frac{1}{n^2} + \frac{1}{n^3} + \frac{1}{n^4} - \cdots;
$$

$$
\frac{1}{(n^2 + n)^2} = \frac{1}{n^4} - \frac{2}{n^5} + \frac{3}{n^6} - \cdots.
$$

So there’s lots of helpful cancellation, and we find

$$
S_n = n^{-1} - \frac{\frac{1}{2} n^{-2} + \frac{1}{3} n^{-3} - \frac{1}{4} n^{-4} + \frac{1}{5} n^{-5} - \frac{1}{6} n^{-6}}{}
$$

$$
- \frac{1}{2} n^{-3} + \frac{1}{3} n^{-4} - \frac{1}{2} n^{-5} + \frac{1}{5} n^{-6}
$$

$$
+ \frac{1}{6} n^{-5} - \frac{1}{4} n^{-6}
$$

plus terms that are $O(n^{-7})$. A bit of arithmetic and we’re home free:

$$
S_n = n^{-1} - \frac{\frac{1}{2} n^{-2} + \frac{1}{3} n^{-3} + \frac{1}{4} n^{-4} = \frac{2}{13} n^{-5} + \frac{1}{12} n^{-6} + O(n^{-7})}. \quad (9.50)
$$

It would be nice if we could check this answer numerically, as we did when we derived exact results in earlier chapters. Asymptotic formulas are harder to verify; an arbitrarily large constant may be hiding in a $O$ term, so any numerical test is inconclusive. But in practice, we have no reason to believe that an adversary is trying to trap us, so we can assume that the unknown $O$-constants are reasonably small. With a pocket calculator we find that

$$
S_4 = \frac{1}{17} + \frac{1}{18} + \frac{1}{19} + \frac{1}{20} = 0.2170107;
$$

and our asymptotic estimate when $n = 4$ comes to

$$
\frac{1}{4} \left( 1 + \frac{1}{4} \left( -\frac{1}{2} + \frac{1}{3} \left( -\frac{1}{6} + \frac{1}{4} \left( 1 + \frac{1}{4} \left( -\frac{2}{13} + \frac{1}{12} \right) \right) \right) \right) \right) = 0.2170125.
$$

If we had made an error of, say, $\frac{1}{12}$ in the term for $n^{-6}$, a difference of $\frac{1}{12} \frac{1}{4096}$ would have shown up in the fifth decimal place; so our asymptotic answer is probably correct.