Problem 5: An infinite sum.

We turn now to an asymptotic question posed by Solomon Golomb [122]: What is the approximate value of

\[ S_n = \sum_{k \geq 1} \frac{1}{kN_n(k)^2} \]  

(9.51)

where \( N_n(k) \) is the number of digits required to write \( k \) in radix \( n \) notation?

First let’s try again for a ballpark estimate. The number of digits, \( N_n(k) \), is approximately \( \log_n k = \log k / \log n \); so the terms of this sum are roughly \( (\log n)^2 / k(\log k)^2 \). Summing on \( k \) gives

\[ \int_2^\infty \frac{dx}{x(\log x)^2} = \frac{1}{\ln 2} \int_2^\infty \frac{1}{1 + \frac{1}{(\log x)^2}} = \frac{1}{\ln 2} \]

Therefore we expect \( S_n \) to be about \( C(\log n)^2 \), for some constant \( C \).

Hand-wave analyses like this are useful for orientation, but we need better estimates to solve the problem. One idea is to express \( N_n(k) \) exactly:

\[ N_n(k) = \left\lfloor \log_n k \right\rfloor + 1 \]  

(9.52)

Thus, for example, \( k \) has three radix \( n \) digits when \( n^2 \leq k < n^3 \), and this happens precisely when \( \left\lfloor \log_n k \right\rfloor = 2 \). It follows that \( N_n(k) > \log_n k \), hence

\[ S_n = \sum_{k \geq 1} 1/kN_n(k)^2 < 1 + (\log n)^2 \sum_{k \geq 2} 1/k(\log k)^2 \]

Proceeding as in Problem 1, we can try to write \( N_n(k) = \log_n k + 0(1) \) and substitute this into the formula for \( S_n \). The term represented here by \( 0(1) \) is always between 0 and 1, and it is about \( \frac{1}{2} \) on the average, so it seems rather well-behaved. But still, this isn’t a good enough approximation to tell us about \( S_n \); it gives us zero significant figures (that is, high relative error) when \( k \) is small, and these are the terms that contribute the most to the sum. We need a different idea.

The key (as in Problem 4) is to use our manipulative skills to put the sum into a more tractable form, before we resort to asymptotic estimates. We can introduce a new variable of summation, \( m = N_n(k) \):

\[ S_n = \sum_{k, m \geq 1} \frac{[m = N_n(k)]}{km^2} \]

\[ = \sum_{k, m \geq 1} \frac{[n^{-1}m \leq k < n^m]}{km^2} \]

\[ = \sum_{m \geq 1} \frac{1}{m^2} (H_{n^m - 1} - H_{n^{m-1} - 1}) \]