This may look worse than the sum we began with, but it’s actually a step forward, because we have very good approximations for the harmonic numbers.

Still, we hold back and try to simplify some more. No need to rush into asymptotics. Summation by parts allows us to group the terms for each value of $H_{n^k-1}$ that we need to approximate:

$$S_n = \sum_{k \geq 1} H_{n^k-1} \left( \frac{1}{k^2} - \frac{1}{(k+1)^2} \right).$$

For example, $H_{n^2-1}$ is multiplied by $1/2^2$ and then by $-1/3^2$. (We have used the fact that $H_{n^0-1} = H_0 = 0$.)

Now we’re ready to expand the harmonic numbers. Our experience with estimating $(n-1)!$ has taught us that it will be easier to estimate $H_{n^k}$ than $H_{n^k+1}$, since the $(n^k - 1)$’s will be messy; therefore we write

$$H_{n^k-1} = H_{n^k} - \frac{1}{n^k} = \ln n^k + \gamma + \frac{1}{2n^k} + O\left(\frac{1}{n^{2k}}\right) - \frac{1}{n^k} = k \ln n + \gamma - \frac{1}{2n^k} + O\left(\frac{1}{n^{2k}}\right).$$

Our sum now reduces to

$$S_n = \sum_{k \geq 1} \left( k \ln n + \gamma - \frac{1}{2n^k} + O\left(\frac{1}{n^{2k}}\right) \right) \left( \frac{1}{k^2} - \frac{1}{(k+1)^2} \right)$$

$$= (\ln n) \Sigma_1 + \gamma \Sigma_2 - \frac{1}{2} \Sigma_3(n) + O\left(\Sigma_3(n^2)\right). \quad (9.53)$$

There are four easy pieces left: $\Sigma_1$, $\Sigma_2$, $\Sigma_3(n)$, and $\Sigma_3(n^2)$.

Let’s do the $\Sigma_3$’s first, since $\Sigma_3(n^2)$ is the 0 term; then we’ll see what sort of error we’re getting. (There’s no sense carrying out other calculations with perfect accuracy if they will be absorbed into a 0 anyway.) This sum is simply a power series,

$$\Sigma_3(x) = \sum_{k \geq 1} \left( \frac{1}{k^2} - \frac{1}{(k+1)^2} \right) x^{-k},$$

and the series converges when $x \geq 1$ so we can truncate it at any desired point.

If we stop $\Sigma_3(n^2)$ at the term for $k = 1$, we get $\Sigma_3(n^2) = O(n^{-2})$; hence (9.53) has an absolute error of $O(n^{-2})$. (To decrease this absolute error, we could use a better approximation to $H_{n^k}$; but $O(n^{-2})$ is good enough for now.) If we truncate $\Sigma_3(n)$ at the term for $k = 2$, we get

$$\Sigma_3(n) = \frac{3}{2} n^{-1} + O(n^{-2});$$

this is all the accuracy we need.