This preliminary analysis indicates that we’ll find it advantageous to write

\[
\phi(n) = \frac{1}{2} \sum_{k=1}^{n} \mu(k) \left( \frac{n}{k} + O(1) \right)^2 = \frac{1}{2} \sum_{k=1}^{n} \mu(k) \left( \frac{n^2}{k^2} + O\left(\frac{n}{k}\right) \right) \\
= \frac{1}{2} \sum_{k=1}^{n} \mu(k)\left(\frac{n^2}{k^2}\right) + \sum_{k=1}^{n} O\left(\frac{n}{k}\right) \\
= \frac{1}{2} \sum_{k=1}^{n} \mu(k)\left(\frac{n^2}{k^2}\right) + O(n \log n)
\]

This removes the floors; the remaining problem is to evaluate the unfloored sum \( \sum_{k=1}^{n} \mu(k)n^2/k^2 \) with an accuracy of \( O(n \log n) \); in other words, we want to evaluate \( \sum_{k=1}^{n} \mu(k)1/k^2 \) with an accuracy of \( O(n^{-1} \log n) \). But that’s easy; we can simply run the sum all the way up to \( k = \infty \), because the newly added terms are

\[
\sum_{k>n} \frac{\mu(k)}{k^2} = O\left(\sum_{k>n} \frac{1}{k^2}\right) = O\left(\sum_{k>1} \frac{1}{k(k-1)}\right) \\
= O\left(\sum_{k>1} \left(\frac{1}{k-1} - \frac{1}{k}\right)\right) = O\left(\frac{1}{n}\right).
\]

We proved in (7.88) that \( \sum_{k \geq 1} \mu(k)/k^2 = 1/\zeta(2) \). Hence \( \sum_{k \geq 1} \mu(k)/k^2 = 1/(\sum_{k \geq 1} 1/k^2) = 6/\pi^4 \), and we have our answer:

\[
\phi(n) = \frac{3}{\pi^2} n^2 + O(n \log n). \tag{9.56}
\]

### 9.4 TWO ASYMPTOTIC TRICKS

Now that we have some facility with \( \theta \) manipulations, let’s look at what we’ve done from a slightly higher perspective. Then we’ll have some important weapons in our asymptotic arsenal, when we need to do battle with tougher problems.

**Trick 1: Boots trapping.**

When we estimated the \( n \)th prime \( P_n \) in Problem 3 of Section 9.3, we solved an asymptotic recurrence of the form

\[
P_n = n \ln P_n (1 + O(1/\log n)).
\]

We proved that \( P_n = n \ln n + O(n) \) by first using the recurrence to show the weaker result \( O(n^2) \). This is a special case of a general method called bootstrapping, in which we solve a recurrence asymptotically by starting with