a rough estimate and plugging it into the recurrence; in this way we can often derive better and better estimates, “pulling ourselves up by our bootstraps.”

Here’s another problem that illustrates bootstrapping nicely: What is the asymptotic value of the coefficient $g_n = [z^n] G(z)$ in the generating function

$$G(z) = \exp\left(\sum_{k \geq 1} \frac{z^k}{k^2}\right), \quad (9.57)$$

as $n \to \infty$? If we differentiate this equation with respect to $z$, we find

$$G'(z) = \sum_{n=0}^{\infty} n g_n z^{n-1} = \left(\sum_{k \geq 1} \frac{z^{k-1}}{k}\right) G(z);$$

equating coefficients of $z^{n-1}$ on both sides gives the recurrence

$$n g_n = \sum_{0 \leq k < n} \frac{g_k}{n-k}. \quad (9.58)$$

Our problem is equivalent to finding an asymptotic formula for the solution to (9.58), with the initial condition $g_0 = 1$. The first few values

<table>
<thead>
<tr>
<th>n</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g_n$</td>
<td>, , $\frac{1}{4}$, $\frac{19}{36}$, $\frac{107}{288}$, $\frac{641}{1100}$, $\frac{51103}{552960}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

don’t reveal much of a pattern, and the integer sequence $\langle n!^2 g_n \rangle$ doesn’t appear in Sloane’s Handbook [270]; therefore a closed form for $g_n$ seems out of the question, and asymptotic information is probably the best we can hope to derive.

Our first handle on this problem is the observation that $0 < g_n \leq 1$ for all $n \geq 0$; this is easy to prove by induction. So we have a start:

$$g_n = O(1),$$

This equation can, in fact, be used to “prime the pump” for a bootstrapping operation: Plugging it in on the right of (9.58) yields

$$n g_n = \sum_{0 \leq k < n} \frac{O(1)}{n-k} = H_n O(1) = O(\log n);$$

hence we have

$$g_n = O\left(\frac{\log n}{n}\right), \quad \text{for } n > 1.$$