And we can bootstrap yet again:

\[
\begin{align*}
\text{n}_g \cdot n &= \frac{1}{n} + \sum_{0 \leq k < n} \frac{O((1 + \log k)/k)}{n-k} \\
&= \frac{1}{n} + \sum_{0 \leq k < n} \frac{O(\log n)}{k(n-k)} \\
&= \frac{1}{n} + \sum_{0 < k < n} \left( \frac{1}{k} + \frac{1}{n-k} \right) O(\log n) \\
&= \frac{1}{n} + \frac{2}{n} H_{n-1} O(\log n) = \frac{1}{n} O(\log n)^2,
\end{align*}
\]

obtaining

\[
g_n = O\left(\frac{\log n}{n}\right)^2,
\]

(9.59)

Will this go on forever? Perhaps we'll have \( g_n = O(n^{-1} \log n)^m \) for all \( m \).

Actually no; we have just reached a point of diminishing returns. The next attempt at bootstrapping involves the sum

\[
\sum_{0 < k < n} \frac{1}{k^2(n-k)} \sum_{0 < k < n} \left( \frac{1}{nk^2} + \frac{1}{n^2 k} + \frac{1}{n^2(n-k)} \right)
\]

\[
\begin{align*}
&= \frac{1}{n} H_n^{(2)} + \frac{2}{n^2} H_{n-1}.
\end{align*}
\]

which is \( \Omega(n^{-1}) \); so we cannot get an estimate for \( g_n \) that falls below \( \Omega(n^{-2}) \).

In fact, we now know enough about \( g_n \) to apply our old trick of pulling out the largest part:

\[
\text{n}_g \cdot n = \sum_{0 < k < n} \frac{g_k}{n} + \sum_{0 < k < n} g_k \left( \frac{1}{n-k} - \frac{1}{n} \right)
\]

\[
= \frac{1}{n} \sum_{k > 0} g_k - \frac{1}{n} \sum_{k \geq n} g_k + \frac{1}{n} \sum_{0 < k < n} \frac{kg_k}{n-k}
\]

(9.60)

The first sum here is \( G(1) = \exp \left( \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \cdots \right) = e^{\pi^2/6} \), because \( G(z) \) converges for all \( |z| \leq 1 \). The second sum is the tail of the first; we can get an upper bound by using (9.59):

\[
\sum_{k \geq n} g_k = O\left( \sum_{k \geq n} \frac{(\log k)^2}{k^2} \right) = O\left( \frac{(\log n)^2}{n} \right).
\]