Here's another example where tail switching is effective. (Unlike our previous examples, this one illustrates the trick in its full generality, with \(E(n) \neq 0\).) We seek the asymptotic value of

\[
L_n = \sum_{k \geq 0} \ln(n + 2k) / k!.
\]

The big contributions to this sum occur when \(k\) is small, because of the \(k!\) in the denominator. In this range we have

\[
\ln(n + 2k) = \ln n + \frac{2^k}{n} - \frac{2^{2k}}{2n^2} + O\left(\frac{2^{3k}}{n^3}\right)
\]

We can prove that this estimate holds for \(0 \leq k < [\lg n]\), since the original terms that have been truncated with 0 are bounded by the convergent series

\[
\sum_{m \geq 3} \frac{2^k m}{m n^m} \leq \frac{2^{3k}}{n^3} \sum_{m \geq 3} \frac{2^k (m-3)}{n^{m-3}} \leq \frac{2^{3k}}{n^3} \left( 1 + \frac{1}{2} + \frac{1}{4} + \cdots \right) = \frac{2^{3k}}{n^3} \cdot 2.
\]

(In this range, \(2^k/n \leq 2^{[\lg n]}^{-1}/n \leq \frac{1}{2}\).) Therefore we can apply the three-step method just described, with

\[
\begin{align*}
ok(n) &= \ln(n + 2k)/k!, \\
bk(n) &= (\ln n + 2^k/n - 4^k/2n^2)/k!, \\
ck(n) &= 8^k/n^3 k!;
\end{align*}
\]

\[
D_n = \{0, 1, \ldots, [\lg n] - 1\}, \\
T_n = \{[\lg n], [\lg n] + 1, \ldots\}.
\]

All we have to do is find good bounds on the three \(\Sigma\)'s in (9.63), and we'll know that \(\Sigma\)'s \(= \sum_{k \geq 0} a_k(n) \approx \sum_{k \geq 0} b_k(n)\).

The error we have committed in the dominant part of the sum, \(\Sigma_c(n) = \sum_{k \in D_n} 8^k/n^3 k!\), is obviously bounded by \(\sum_{k \geq 0} 8^k/n^3 k! = e^{8/n^3}\), so it can be replaced by \(O(1)\). The new tail error is

\[
|\Sigma_b(n)| = \left| \sum_{k \geq [\lg n]} b_k(n) \right| < \sum_{k \geq [\lg n]} \ln n + 2^k + 4^k / k! < \frac{\ln n + 2^k + 4^k}{[\lg n]!} \sum_{k \geq 0} \frac{4^k}{k!} = O\left(\frac{n^2}{[\lg n]}\right).
\]