On the other hand, for the Euler method, the change in $y_{n+1}$ in Eq. (13) due to $\delta$ is $\delta(1 + rh)$. If we recall that $r \leq 0$ and require that $|1 + rh| \leq 1$, then we find that $h$ must satisfy $h \leq 2/|r|$. Thus the Euler method is not stable for this problem unless $h$ is sufficiently small.

The restriction on the step size $h$ in using the Euler method in the preceding example is rather mild unless $|r|$ is quite large. Nonetheless, the example illustrates that it may be necessary to restrict $h$ in order to achieve stability in the numerical method, even though the initial value problem itself is stable for all values of $h$. Problems for which a much smaller step size is needed for stability than for accuracy are called stiff. The backward differentiation formulas described in Section 8.4 (of which the backward Euler formula is the lowest order example) are the most popular formulas for solving stiff problems. The following example illustrates the kind of instabilities that can occur in trying to approximate the solution of a stiff problem.

Consider the initial value problem

$$y' = -100y + 100t + 1, \quad y(0) = 1. \quad (16)$$

Find numerical approximations to the solution for $0 \leq t \leq 1$ using the Euler, backward Euler, and Runge–Kutta methods. Compare the numerical results with the exact solution.

Since the differential equation is linear, it is easy to solve, and the solution of the initial value problem (16) is

$$y = \phi(t) = e^{-100t} + t. \quad (17)$$

Some values of the solution $\phi(t)$, correct to six decimal places, are given in the second column of Table 8.5.3 and a graph of the solution is shown in Figure 8.5.2. There is a thin layer (sometimes called a boundary layer) to the right of $t = 0$ in which the exponential term is significant and the solution varies rapidly. Once past this layer, however, $\phi(t) \equiv t$ and the graph of the solution is essentially a straight line. The width of the boundary layer is somewhat arbitrary, but it is certainly small. At $t = 0.1$, for example, $\exp(-100t) \approx 0.000045$.

![FIGURE 8.5.2 The solution of the initial value problem (16).](image-url)