Since \([\log n]!\) grows faster than any power of \(n\), this minuscule error is overwhelmed by \(\Sigma c(n) = O(n^{-3})\). The error that comes from the original tail,

\[
\Sigma_a(n) = \sum_{k \geq [\log n]} a_k(n) < \sum_{k \geq [\log n]} \frac{k + \log n}{k!},
\]

is smaller yet.

Finally, it’s easy to sum \(\sum_{k \geq 0} b_k(n)\) in closed form, and we have obtained the desired asymptotic formula:

\[
\sum_{k \geq 0} \frac{\log(n + 2^k)}{k!} = e \log n + \frac{e^2}{n} - \frac{e^4}{2n^2} + O\left(\frac{1}{n^3}\right).
\]

The method we’ve used makes it clear that, in fact,

\[
\sum_{k \geq 0} \frac{\log(n + 2^k)}{k!} = e \log n + \sum_{k=1}^{m-1} (-1)^{k+1} \frac{e^{2^k}}{k n^k} + O\left(\frac{1}{n^m}\right),
\]

for any fixed \(m > 0\). (This is a truncation of a series that diverges for all fixed \(n\) if we let \(m \to \infty\).)

There’s only one flaw in our solution: We were too cautious. We derived (9.64) on the assumption that \(k < [\log n]\), but exercise 53 proves that the stated estimate is actually valid for all values of \(k\). If we had known the stronger general result, we wouldn’t have had to use the two-tail trick; we could have gone directly to the final formula! But later we’ll encounter problems where exchange of tails is the only decent approach available.

### 9.5 EULER’S SUMMATION FORMULA

And now for our next trick—which is, in fact, the last important technique that will be discussed in this book—we turn to a general method of approximating sums that was first published by Leonhard Euler [82] in 1732. (The idea is sometimes also associated with the name of Colin Maclaurin, a professor of mathematics at Edinburgh who discovered it independently a short time later [211, page 305].)

Here’s the formula:

\[
\sum_{a \leq k < b} f(k) = \int_{a}^{b} f(x) \, dx + \sum_{k=1}^{m} \frac{B_k}{k!} f^{(k-1)}(x) \bigg|_{a}^{b} + R_m, \quad (9.67)
\]

where \(R_m = (-1)^{m+1} \int_{a}^{b} \frac{B_m}{m!} f^{(m)}(x) \, dx\) for integers \(a \leq b;\)

\[
\text{integer } m \geq 1. \quad (9.68)
\]