On the left is a typical sum that we might want to evaluate. On the right is another expression for that sum, involving integrals and derivatives. If $f(x)$ is a sufficiently “smooth” function, it will have $m$ derivatives $f'(x), \ldots, f^{(m)}(x)$, and this formula turns out to be an identity. The right-hand side is often an excellent approximation to the sum on the left, in the sense that the remainder $R_m$ is often small. For example, we’ll see that Stirling’s approximation for $n!$ is a consequence of Euler’s summation formula; so is our asymptotic approximation for the harmonic number $H_n$.

The numbers $B_k$ in (9.67) are the Bernoulli numbers that we met in Chapter 6; the function $B_m([x])$ in (9.68) is the Bernoulli polynomial that we met in Chapter 7. The notation $\{x\}$ stands for the fractional part $x - \lfloor x \rfloor$, as in Chapter 3. Euler’s summation formula sort of brings everything together.

Let’s recall the values of small Bernoulli numbers, since it’s always handy to have them listed near Euler’s general formula:

$$
B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = -\frac{1}{30};
$$

$$
B_3 = B_5 = B_7 = B_9 = B_{11} = \cdots = 0.
$$

Jakob Bernoulli discovered these numbers when studying the sums of powers of integers, and Euler’s formula explains why: If we set $f(x) = x^{m-1}$, we have $f^{(m)}(x) = 0$; hence $R_m = 0$, and (9.67) reduces to

$$
\sum_{a \leq k < b} k^{m-1} = \frac{x^m}{m} \bigg|_a^b + \sum_{k=1}^{m} \frac{B_k}{k!} (m-1)! (x - \lfloor x \rfloor)^{m-k} \bigg|_a^b
$$

$$
= \frac{1}{m} \sum_{k=0}^{m} \binom{m}{k} B_k \cdot (b^{m-k} - a^{m-k}).
$$

For example, when $m = 3$ we have our favorite example of summation:

$$
\sum_{0 \leq k < n} k^2 = \frac{1}{3} \left( \binom{3}{0} B_0 n^3 + \binom{3}{1} B_1 n^2 + \binom{3}{2} B_2 n \right) = \frac{n^3}{3} - \frac{n^2}{2} + \frac{n}{6}
$$

(This is the last time we shall derive this famous formula in this book.)

Before we prove Euler’s formula, let’s look at a high-level reason (due to Lagrange [192]) why such a formula ought to exist. Chapter 2 defines the difference operator $A$ and explains that $\sum$ is the inverse of $A$, just as $\int$ is the inverse of the derivative operator $D$. We can express $A$ in terms of $D$ using Taylor’s formula as follows:

$$
f(x + \varepsilon) = f(x) + \frac{f'(x)}{1!} \varepsilon + \frac{f''(x)}{2!} \varepsilon^2 + \cdots.
$$