Setting \( c = 1 \) tells us that

\[
Af(x) = f(x + 1) - f(x) = f'(x)/1! + f''(x)/2! + f'''(x)/3! + \cdots \\
= (D/1! + D^2/2! + D^3/3! + \cdots) f(x) = (e^D - 1) f(x). \quad (9.69)
\]

Here \( e^D \) stands for the differential operation \( 1 + D/1! + D^2/2! + D^3/3! + \ldots \).

Since \( A = e^D - 1 \), the inverse operator \( \Sigma = 1/A \) should be \( 1/(e^D - 1) \); and we know from Table 337 that \( z/(e^z - 1) = \sum_{k \geq 0} B_k z^k/k! \) is a power series involving Bernoulli numbers. Thus

\[
\sum = \frac{B_0}{D} + \frac{B_1}{1!} + \frac{B_2}{2!} D + \frac{B_3}{3!} D^2 + \cdots = \int + \sum_{k \geq 1} \frac{B_k D^{k-1}}{k!}. \quad (9.70)
\]

Applying this operator equation to \( f(x) \) and attaching limits yields

\[
\sum_a^b f(x) \, dx = \int_a^b f(x) \, dx + \sum_{k \geq 1} \frac{B_k}{k!} f^{(k-1)}(x) \bigg|_a^b,
\]

which is exactly Euler’s summation formula \((9.67)\) without the remainder term. (Euler did not, in fact, consider the remainder, nor did anybody else until S. D. Poisson [236] published an important memoir about approximate summation in 1823. The remainder term is important, because the infinite sum \( \sum_{k \geq 1} (B_k/k!) f^{(k-1)}(x) \big|_a^b \) often diverges. Our derivation of \((9.71)\) has been purely formal, without regard to convergence.)

Now let’s prove \((9.67)\), with the remainder included. It suffices to prove the case \( a = 0 \) and \( b = 1 \), namely

\[
f(0) = \int_0^1 f(x) \, dx + \sum_{k \geq 1} \frac{B_k}{k!} f^{(k-1)}(x) \bigg|_0^1 = (-1)^m \int_0^1 \frac{B_m(x)}{m!} f^{(m)}(x) \, dx,
\]

because we can then replace \( f(x) \) by \( f(x + 1) \) for any integer \( 1 \), getting

\[
f(1) = \int_0^{1+} f(x) \, dx + \sum_{k \geq 1} \frac{B_k}{k!} f^{(k-1)}(x) \bigg|_0^{1+} = (-1)^{m+1} \int_0^{1+} \frac{B_m(x)}{m!} f^{(m)}(x) \, dx
\]

The general formula \((9.67)\) is just the sum of this identity over the range \( a \leq 1 < b \), because intermediate terms telescope nicely.

The proof when \( a = 0 \) and \( b = 1 \) is by induction on \( m \), starting with \( m = 1 \):

\[
f(0) = \int_0^1 f(x) \, dx - \frac{1}{2} (f(1) - f(0)) + \int_0^1 (x - \frac{1}{2}) f'(x) \, dx.
\]