The Bernoulli polynomial $B_m(x)$ is defined by the equation

$$B_m(x) = \binom{m}{0} B_0 x^m + \binom{m}{1} B_1 x^{m-1} + \ldots + \binom{m}{m} B_m x^0$$  \hspace{1cm} (9.72)

in general, hence $B_1(x) = x - \frac{1}{2}$ in particular. In other words, we want to prove that

$$\frac{f(0) + f(1)}{2} = \int_0^1 f(x) \, dx + \int_0^1 (x - \frac{1}{2}) f'(x) \, dx.$$  \hspace{1cm} (9.73)

But this is just a special case of the formula

$$u(x)v(x) \bigg|_0^1 = \int_0^1 u(x) \, dv(x) + \int_0^1 v(x) \, du(x)$$  \hspace{1cm} (9.74)

for integration by parts, with $u(x) = f(x)$ and $v(x) = x - \frac{1}{2}$. Hence the case $m = 1$ is easy.

To pass from $m = 1$ to $m$ and complete the induction when $m > 1$, we need to show that $R_{m-1} = (B_m/m!) f^{(m-1)}(x) \big|_0^1 + R_m$, namely that

$$\left( -1 \right)^m \int_0^1 \frac{B_{m-1}(x)}{(m-1)!} f^{(m-1)}(x) \, dx
= \frac{B_m}{m!} f^{(m-1)}(x) \big|_0^1 - \left( -1 \right)^m \int_0^1 \frac{B_m(x)}{m!} f^{(m)}(x) \, dx.$$  \hspace{1cm} (9.75)

This reduces to the equation

$$\left( -1 \right)^m B_m f^{(m-1)}(x) \big|_0^1 = m \int_0^1 B_{m-1}(x) f^{(m-1)}(x) \, dx + \int_0^1 B_m(x) f^{(m)}(x) \, dx.$$  \hspace{1cm} (9.76)

Once again (9.73) applies to these two integrals, with $u(x) = f^{(m-1)}(x)$ and $v(x) = B_m(x)$, because the derivative of the Bernoulli polynomial (9.72) is

$$\frac{d}{dx} \sum_k \binom{m}{k} B_k x^{m-k} = \sum_k \binom{m}{k} (m-k) B_k x^{m-k-1}
= m \sum_k \binom{m-1}{k} B_k x^{m-1-k} = m B_{m-1}(x).$$  \hspace{1cm} (9.77)

(The absorption identity (5.7) was useful here.) Therefore the required formula will hold if and only if

$$\left( -1 \right)^m B_m f^{(m-1)}(x) \big|_0^1 = B_m(x) f^{(m-1)}(x) \big|_0^1.$$

Will the authors never get serious?