In other words, we need to have

\[ (-l)^m B_m = B_m(1) = B_m(0), \quad \text{for } m > 1. \tag{9.75} \]

This is a bit embarrassing, because \( B_m(0) \) is obviously equal to \( B_m \), not to \( (-l)^m B_m \). But there’s no problem really, because \( m > 1 \); we know that \( B_m \) is zero when \( m \) is odd. (Still, that was a close call.)

To complete the proof of Euler’s summation formula we need to show that \( B_m(1) = B_m(0) \), which is the same as saying that

\[ \sum_k \binom{m}{k} B_k = B_m, \quad \text{for } m > 1. \]

But this is just the definition of Bernoulli numbers, \( (6.79) \), so we’re done.

The identity \( B_m'(x) = mB_{m-1}(x) \) implies that

\[ \int_0^1 B_m(x) \, dx = \frac{B_{m+1}(1) - B_{m+1}(0)}{m+1}, \]

and we know now that this integral is zero when \( m \geq 1 \). Hence the remainder term in Euler’s formula,

\[ R_m = \frac{(-l)^{m+1}}{m!} \int_a^b B_m((x)) f^{(m)}(x) \, dx, \]

multiplies \( f^{(m)}(x) \) by a function \( B_m((x)) \) whose average value is zero. This means that \( R_m \) has a reasonable chance of being small.

Let’s look more closely at \( B_m(x) \) for \( 0 \leq x \leq 1 \), since \( B_m(x) \) governs the behavior of \( R_m \). Here are the graphs for \( B_m(x) \) for the first twelve values of \( m \):

\[
\begin{align*}
  &m = 1 & m = 2 & m = 3 & m = 4 \\
  B_m(x) & & & & \\
  B_{4+m}(x) & & & & \\
  B_{8+m}(x) & & & & \\
\end{align*}
\]

Although \( B_3(x) \) through \( B_6(x) \) are quite small, the Bernoulli polynomials and numbers ultimately get quite large. Fortunately \( R_m \) has a compensating factor \( \frac{1}{m!} \), which helps to calm things down.