The graph of \( B_n(x) \) begins to look very much like a sine wave when \( m \geq 3 \); exercise 58 proves that \( B_n(x) \) can in fact be well approximated by a negative multiple of \( \cos(2\pi x - \pi m) \), with relative error \( 1/2^m \).

In general, \( B_{4k+1}(x) \) is negative for \( 0 < x < \frac{1}{2} \) and positive for \( \frac{1}{2} < x < 1 \). Therefore its integral, \( B_{4k+2}(x)/(4k+2) \), decreases for \( 0 < x < \frac{1}{2} \) and increases for \( \frac{1}{2} < x < 1 \). Moreover, we have

\[
B_{4k+1}(1 - x) = -B_{4k+1}(x), \quad \text{for } 0 \leq x \leq 1,
\]

and it follows that

\[
B_{4k+2}(1 - x) = B_{4k+2}(x), \quad \text{for } 0 \leq x \leq 1.
\]

The constant term \( B_{4k+2} \) causes the integral \( \int_0^1 B_{4k+2}(x) \, dx \) to be zero; hence \( B_{4k+2} > 0 \). The integral of \( B_{4k+2}(x) \) is \( B_{4k+3}(x)/(4k+3) \), which must therefore be positive when \( 0 < x < \frac{1}{2} \) and negative when \( \frac{1}{2} < x < 1 \); furthermore \( B_{4k+3}(1 - x) = -B_{4k+3}(x) \), so \( B_{4k+3}(x) \) has the properties stated for \( B_{4k+1}(x) \), but negated. Therefore \( B_{4k+4}(x) \) has the properties stated for \( B_{4k+2}(x) \), but negated. Therefore \( B_{4k+5}(x) \) has the properties stated for \( B_{4k+1}(x) \); we have completed a cycle that establishes the stated properties inductively for all \( k \).

According to this analysis, the maximum value of \( B_{2m+1}(x) \) must occur either at \( x = 0 \) or at \( x = \frac{1}{2} \). Exercise 17 proves that

\[
B_{2m}(\frac{1}{2}) = (2^{1-2m} - 1)B_{2m}.
\]

hence we have

\[
|B_{2m}(x)| \leq |B_{2m}|.
\]

This can be used to establish a useful upper bound on the remainder in Euler's summation formula, because we know from (6.89) that

\[
\frac{|B_{2m}|}{(2m)!} = \frac{2}{(2\pi)^{2m}} \sum_{k \geq 1} \frac{1}{k^{2m}} = O\left(2\pi^{-2m}\right), \quad \text{when } m > 0.
\]

Therefore we can rewrite Euler's formula (9.67) as follows:

\[
\sum_{a \leq k < b} f(k) = \int_a^b f(x) \, dx - \frac{1}{2} f(x)^|b| + \sum_{k=1}^{m} \left[ \frac{B_{2k}}{(2k)!} f^{(2k-1)}(x) \right]^b_a + O\left((2\pi)^{-2m}\right) \int_a^b f^{(2m)}(x) \, dx.
\]

For example, if \( f(x) = e^x \), all derivatives are the same and this formula tells us that

\[
\sum_{a \leq k < b} e^k = (e^b - e^a)(1 - \frac{1}{2} + B_{2}/2! + B_{4}/4! + \cdots + B_{2m}/(2m)!).
\]